22. *On the Telegraph Equation and the Toda Equation*

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1984)

§1. **Summary.** We can solve the Toda equation with two time variables

(1.1) $$XY \log t_n = t_{n+1}t_{n-1}/t_n^2$$

($X=\partial/\partial x$, $Y=\partial/\partial y$, $t_n=t_n(x,y)$) using solutions of the telegraph equation

(1.2) $$(XY+1)u_n=0.$$

Rational solutions, Bessel function solutions and solutions which are expressed by hypergeometric functions with two variables are obtained.

§2. **Bäcklund transformation.** When t_n satisfies (1.1)

(2.1) $$r_n=XY\log t_n, \qquad s_n=Y\log t_{n-1}/t_n$$

satisfies

(2.2) $$Yr_n=r_n(s_n-s_{n+1}), \qquad Xs_n=r_{n-1}-r_n.$$

Let us introduce the following triple of partial differential operators

(2.3) $$M_n=XY+s_{n+1}X+r_n, \quad X_n=-r_n^{-1}X, \quad Y_n=Y+s_{n+1}.$$

Define

(2.4) $$T=\{u_n\,;\, M_0u_0=0,\ u_{n+1}=Y_nu_n\ (n\geqslant 0),\ u_{n-1}=X_nu_n\ (n\leqslant 0)\}.$$

We can show

Theorem 2.1 (Bäcklund transformation). *If* $u_n\in T$ *then we have* $M_nu_n=0$, $u_{n+1}=Y_nu_n$, $u_{n-1}=X_nu_n$ ($n=0, \pm 1, \pm 2, \cdots$) *and* $\tau_n=u_nt_n$ *satisfies the Toda equation* (1.1).

We can obtain all solutions of the Toda equation (2.2) with separated form $r_n=f(n)g(x,y)$. $f(n)$ must be a polynomial in n of order 2 and our solutions are

(i) $r_n=(n-\alpha)(n-\beta)a'(x)b'(y)(a(x)+b(y))^{-2}$,

(ii) $r_n=(n-\alpha)a(x)b(y)$, (iii) $r_n=a(x)b(y)$,

where α and β are arbitrary constants and $a(x)$ and $b(y)$ are arbitrary functions. In this note we only treat the Bäcklund transforms of the simplest solutions (iii).

§3. **One-parameter groups on *T*.** No loss of generality we can assume that $a(x)=b(y)=1$. In this case we have

(3.1) $$t_n=e^{xy}, \quad r_n=1, \quad s_n=0,$$

(3.2) $$M_n=M=XY+1, \quad X_n=-X, \quad Y_n=Y.$$

We can determine all of the first order partial differential operators $D=a(x,y)X+b(x,y)Y+c(x,y)$ which commute with M (modulo M).