15. On the Essential Spectrum of MHD Plasma in Toroidal Region

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1. Introduction. Related to the plasma confinement problem, the following second order differential equation is of some interest:

$$(1.1)\quad \begin{aligned}\rho(\partial^2\xi/\partial t^2)&=\mathrm{grad}\,\{\gamma P(\mathrm{div}\,\xi)+(\mathrm{grad}\,P)\cdot\xi\}\\&\quad+(1/\mu)\{B\times\mathrm{rot}\,(\mathrm{rot}\,(B\times\xi))-(\mathrm{rot}\,B)\times\mathrm{rot}\,(B\times\xi)\}\\&\equiv-\rho^{1/2}K\rho^{1/2}\xi.\end{aligned}$$

Here, $\xi(t,r)$ is related to the velocity field $V(t,r)$ as $d\xi/dt=V(t,\xi+r)$, $\xi(0,r)=0$, and is called the Lagrangian displacement vector. The quantities ρ, P and B are independent of t and are the solutions of the plasma equilibrium satisfying:

$$(1.2)\quad \mathrm{grad}\,P=j\times B,\quad j=(1/\mu)\,\mathrm{rot}\,B,\quad \mathrm{div}\,B=0,$$

with P, $\rho\geqq\varepsilon_0>0$. Further, (1.1) is derived from the following magnetohydrodynamic (MHD in short) system:

$$(1.3)\quad \begin{cases}\dfrac{\partial\rho}{\partial t}+\mathrm{div}\,(\rho V)=0,\quad \dfrac{D}{Dt}(P\rho^{-\gamma})=0,\quad \rho\dfrac{DV}{Dt}=-\mathrm{grad}\,P+j\times B,\\[2mm] \dfrac{\partial B}{\partial t}=-\mathrm{rot}\,E,\quad \mathrm{div}\,B=0,\quad E+V\times B=0,\quad j=\dfrac{1}{\mu}\,\mathrm{rot}\,B,\end{cases}$$

by means of the linearization in the vicinity of the equilibrium (1.2). Here, ρ, P, V and j are respectively the density, the pressure, the velocity and the electric current density of the plasma, and B and E are the magnetic and electric fields, and μ is the permeability and γ is the specific heat ratio, and $D/Dt=\partial/\partial t+V\cdot\mathrm{grad}$ is the convective derivative.

In the following, we shall investigate the spectral properties of K. Especially, we consider (1.1) in the axisymmetric toroidal region Ω in R^3 and around the following special axisymmetric equilibrium (cf. Temam [5], Friedman [2] §§14–18). Namely, Ω is defined as

$$\Omega=\{r=(x,y,z)\,|\,a_1<\psi(r,\vartheta,z)<a_2,\ x=r\cos\vartheta,\ y=r\sin\vartheta\},$$

where $\psi=\psi(r,z)$ with $r=(x^2+y^2)^{1/2}$ satisfies the non-linear elliptic differential equation (Grad-Shafranov equation):

$$-\left(r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\psi=r^2\{\partial P/\partial\psi\}+I\{\partial I/\partial\psi\}$$

with given functions P and I of ψ. In this case, B is given as

$$B=\left(\frac{1}{r}\frac{\partial\psi}{\partial z},\frac{1}{r}I,-\frac{1}{r}\frac{\partial\psi}{\partial r}\right).$$