14. A Remark on the Global Markov Property for the d-Dimensional Ising Model

By Yasunari HIGUCHI
Department of Mathematics, Faculty of Science, Kobe University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1984)

1. **Introduction.** Let Z^d be the d-dimensional cubic lattice and $\Omega\equiv\{-1,+1\}^{Z^d}$ be the configuration space, equipped with the product of discrete topology. $\mathcal{F}$ stands for the Borel σ-field of Ω. The sub σ-fields $\{\mathcal{F}_V\,;\ V\subset Z^d\}$ are defined by

$$\mathcal{F}_V\equiv\sigma\{\omega(x)\,;\ x\in V\}.$$

A probability measure μ on $(\Omega,\mathcal{F})$ is said to have *local Markov property* (LMP), if for every finite $V\subset Z^d$,

$$(1)\qquad \mu(\cdot\,|\mathcal{F}_{V^c})(\omega)=\mu(\cdot\,|\mathcal{F}_{\partial V})(\omega)\qquad \text{on } \mathcal{F}_V\ \mu\text{-a.s. } \omega,$$

where $\partial V\equiv\{x\in V^c\,;\ |x-y|\equiv\max\{|x^i-y^i|\,;\ 1\leqq i\leqq d\}=1 \text{ for some } y\in V\}$. If (1) holds for any $V\subset Z^d$, then μ is said to have *global Markov property* (GMP). It is known that (LMP) does not necessarily imply (GMP) (see for example, [4], [6], [7]). Therefore the question is when (LMP) implies (GMP). In this note, we discuss this question for the d-dimensional Ising model. The Hamiltonian of this model is given for each finite $V\subset Z^d$, with magnetic field h, and the boundary condition $\omega\in\Omega$, by

$$(2)\qquad E_V(\eta\,|\,\omega)=\sum\nolimits_{x,y\in V}J_{x,y}\eta(x)\eta(y)+\sum\nolimits_{x\in V}\sum\nolimits_{y\in\partial V}J_{x,y}\eta(x)\omega(y)+h\sum\nolimits_{x\in V}\eta(x),$$

where $J_{x,y}=J_{0,|x-y|}=0$ unless $|x-y|=1$. For $\beta>0$, the corresponding finite Gibbs state for (2) is given by

$$(3)\qquad P_{\beta,V}(\{\eta(x),\ x\in V\}\,|\,\omega)=(\text{normalization})\cdot\exp\{-\beta E_V(\eta\,|\,\omega)\}.$$

and

$$(4)\qquad P_{\beta,V}(\{\eta(x)=\omega(x),\ x\in V^c\}\,|\,\omega)=1.$$

A Gibbs state for the Ising model (2) is a probability measure μ on $(\Omega,\mathcal{F})$ satisfying

$$(5)\qquad \mu(\cdot\,|\mathcal{F}_{V^c})(\omega)=P_{\beta,V}(\cdot\,|\,\omega)\ \mu\text{-a.s. } \omega,\qquad \text{for every finite } V\subset Z^d.$$

By definition, any Gibbs state for Ising model (2) has (LMP), but not every Gibbs state for (2) has (GMP) (a counterexample is given in [4]). If $J=\{J_{x,y}\}$ satisfies Dobrushin's uniqueness condition, then the unique Gibbs state has (GMP) ([2], [3]).

In this note, we assume that the Ising model (2) has attractive interaction; $J_{x,y}\leqq 0$ for every pair $x,y\in Z^d$. In this case, it is known that there exists a critical β_c, $0<\beta_c\leqq\infty$ (the last equality holds iff $d=1$), such that Gibbs state is unique for $\beta<\beta_c$, and non-unique for