2. A Generalization of Liapunov's Theorem Concerning a Mass of Fluid with Self-Gravitation*)

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§ 1. **Introduction.** Suppose that a mass of fluid with uniform density lies in $\boldsymbol{R}^3$ and that no force other than the gravitational one due to itself acts on it. Then it is intuitively clear that the fluid attains its equilibrium by forming a sphere. M. A. Liapunov proves in [1] that the sphere is the only stable equilibrium figure of the fluid (see also Poincaré [2]). In the present paper we show that among all the figures of the mass of fluid (stable or not) the sphere is the only possible equilibrium figure. Our proof is completely different from Liapunov's. Actually our method is Serrin's moving plane method ([6]).

Remark. We consider only the gravitational force. In particular, the fluid lies still without rotation. In the case where the fluid rotates with a uniform angular velocity, various kinds of equilibrium figures are known to occur to form bifurcations (see [1]–[3], [7], [8]).

§ 2. **Mathematical formulation of the problem.** Let Ω be the domain occupied by the fluid. Suppose that Ω is a bounded connected open set in $\boldsymbol{R}^3$ with a boundary Γ of C^1-class. The density of the fluid is assumed to be unity. We denote by V the potential of the gravitational force vanishing at the infinity. Then V is given by

$$V(x)=\int_\Omega \frac{dy}{|x-y|},$$

if the scales are suitably chosen. The function V is characterized by $V \in C^1(\boldsymbol{R}^3)$ and

(1) $$-\Delta V=4\pi \quad \text{in } \Omega,$$

(2) $$-\Delta V=0 \quad \text{in } \boldsymbol{R}^3\backslash\Omega,$$

(3) $$V(x)\to 0 \quad \text{as } |x|\to\infty.$$

The equation of motion is easily integrated to yield $P=V+\text{constant}$, where P is the pressure. Consequently the equilibrium state is represented by

(4) $$V=\text{constant} \quad \text{on } \Gamma.$$

Hence our goal is to show the following

Theorem 1. *If $V \in C^1(\boldsymbol{R}^3)$ satisfies* (1)–(4), *then Γ is necessarily a sphere.*

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