109. On the Power Semigroup of the Group of Integers

By Takayuki TAMURA
Department of Mathematics, University of California,
Davis, CA 95616 U.S.A.

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1984)

If $G(\cdot)$ is a group, the power semigroup $\mathcal{P}(G)$ is the semigroup of all nonempty subsets of G with respect to the operation defined by $AB=\{ab : a\in A,\ b\in B\}$ for all $A, B\in\mathcal{P}(G)$. The author and Shafer [5] obtained the group of units of $\mathcal{P}(G)$, and Putcha [4] studied the greatest semilattice decomposition of $\mathcal{P}(G)$ of a finite group G, but we know little about archimedean components of $\mathcal{P}(G)$ of an infinite group G.

Let Z be the group of integers under addition and Z_+ the subsemigroup of positive integers. The operation in $\mathcal{P}(Z)$ is denoted by $X+Y=\{x+y : x\in X,\ y\in Y\}$. For $X\in\mathcal{P}(Z)$ and $m\in Z_+$, we let $mX=\underbrace{X+\cdots+X}_{m}$ and $[a,b]=\{z\in Z : a\leq z\leq b\}$ if $a, b\in Z$ with $a\leq b$. For undefined terminology and basic information on commutative semigroups, the reader should refer to [1], [3].

Let $\mathcal{P}^*(Z)$ denote the subsemigroup of $\mathcal{P}(Z)$ consisting of all finite nonempty subsets of Z. If $X\in\mathcal{P}^*(Z)$, the archimedean component of $\mathcal{P}(Z)$ containing X coincides with that of $\mathcal{P}^*(Z)$ containing X. Let $\mathcal{A}\{0,1\}$ denote the archimedean component of $\mathcal{P}(Z)$ containing the element $\{0,1\}$. The purpose of this paper is to investigate the structure of $\mathcal{A}\{0,1\}$.

Let $X=\{x_1, x_2, \cdots, x_k\}\in\mathcal{P}^*(Z)$ where $x_1<x_2<\cdots<x_k$. We define $\min X=x_1$, $\max X=x_k$, $\mathrm{id}(X)=x_2-x_1$, $\mathrm{fd}(X)=x_k-x_{k-1}$, and $\mathrm{md}(X)=\max\{x_2-x_1, \cdots, x_k-x_{k-1}\}$. Note $\mathrm{md}(X)\geqq 1$ unless X is a singleton. If $\mathrm{md}(X)=1$, i.e. $X=[x_1, x_k]$, then X is called *consecutive*. If $\mathrm{id}(X)=\mathrm{fd}(X)=1$, X is called *semi-consecutive*. The following is a main theorem in this paper.

Theorem 1. *Let $X\in\mathcal{P}(Z)$. The following are equivalent*:

(1.1) $X\in\mathcal{A}\{0,1\}$.

(1.2) $nX=\{0,1\}+Y$ *for some* $n\in Z_+$ *and some* $Y\in\mathcal{P}(Z)$.

(1.3) $nX=m\{0,1\}+b$ *for some* $n, m\in Z_+$ *and some* $b\in Z$.

(1.4) X *is semi-consecutive.*

(1.5) nX *is consecutive for some* $n\in Z_+$.

Proof. (1.1)→(1.2) is obvious by archimedeaness.

(1.2)→(1.4). If $X=\{x_1, x_2, \cdots, x_k\}$, $\min(nX)=nx_1$ and the second element of nX is $(n-1)x_1+x_2$. This implies $\mathrm{id}(nX)=\mathrm{id}(X)$. Similarly