98. A New Proof of the Schiffer's Identities on Planar Riemann Surfaces*)

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Introduction. Let D be a domain in the complex plane, bounded by a finite number of curves. For a point z_0 in D, there exist two slit functions $P_j(z)$ ($j=0,1$) on D such that (i) $P_j(z)-(z-z_0)^{-1}$ are regular and vanish at z_0, (ii) $P_0(z)(P_1(z))$ maps D conformally to a domain whose boundary consists of horizontal (vertical) slits. Setting $\Phi_+ =(P_0+P_1)/2$ and $\Phi_-=(P_0-P_1)/2$, Schiffer [9] found that Φ_+ becomes schlicht and the following remarkable identities hold;

$$(*) \qquad \Phi_-(z)=-\frac{1}{\pi}\iint_A \frac{d\xi d\eta}{\zeta-\Phi_+(z)} \qquad \text{for } z\in D,\quad \zeta=\xi+i\eta$$

$$(**) \qquad \iint_A \frac{d\xi d\eta}{(\zeta-w)^2}=0 \qquad \text{for } w\in \mathring{A}$$

where $\mathring{A}$ is the interior of the complement A of the image $\Phi_+(D)$ and the integral in (**) is the Cauchy's principal value. Recently, Burbea showed an application of relation (*) ([2]) and gave another proof of (*) ([3]). He states there without proof that (*) remains valid for general plane domain by using its exhaustion. However, for this proof some detailed studies must be necessary, as the set A varies with the exhaustion.

The main purpose of this note is to give a new direct proof of (*) and (**) for the (generalized) slit functions on an arbitrary planar Riemann surface. Our approach is essentially different from the known one, actually the proof is based on the Hilbert transform and the extremal properties of Φ_+ and Φ_- defined by those slit functions.

1. Preliminaries. For an arbitrary Riemann surface R, let $\Gamma_a=\Gamma_a(R)$ be the Hilbert space of square integrable analytic differentials ω on R with norm $\|\omega\|=\left(\iint_R \omega\wedge *\bar{\omega}\right)^{1/2}$, and $\Gamma_{ase}=\{\omega\in\Gamma_a \mid \omega$ is semiexact, i.e. $\int_\gamma \omega=0$ for every closed curve γ dividing $R\}$. Let q be a point of R and ζ be a fixed local parameter at q with $\zeta(q)=0$. For $\omega\in\Gamma_a$ we denote by $\omega^{(n)}(q)$ the n-th derivative $\omega(\zeta)$ at $\zeta=0$, where $\omega=\omega(\zeta)d\zeta$. Now it is known (cf. [4], [5], also [8], [10]) that for every integer $n\geq 1$ there exist uniquely the *semiexact canonical* (mero-

*) Dedicated to Professor Sigeru Mizohata on his 60th birthday.