95. On Some Euler Products. I

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§1. **Prime sets.** We say that a set P is a "prime set" if P is a countable infinite set having a real valued "norm function" $N: P \to \mathbf{R}$ satisfying the following: (1) $N(p)>1$ for all $p \in P$, and (2) $N(p_i) \to \infty$ as $i \to \infty$ for an (i.e., any) ordering $P=\{p_1, p_2, \cdots\}$. Put $\pi(t, P)=\sharp\{p \in P; N(p) \leqq t\}$ for $t>0$ where $\sharp$ denotes the cardinality. Then, (2) is equivalent to that $\pi(t, P)$ is finite for each $t>0$. We define $d(P)=\inf\{d>0; \sum_p N(p)^{-d}<\infty\}$. Then $0 \leqq d(P) \leqq \infty$, and we have

$$d(P)=\limsup_{t\to\infty} \frac{\log \pi(t, P)}{\log t}.$$

We are exclusively interested in the case of finite $d(P)$, and we define the zeta function of P by $\zeta(s, P)=\prod_p (1-N(p)^{-s})^{-1}$ for a variable s in the complex numbers $\mathbf{C}$. This infinite product (an Euler product over P) converges absolutely in $\mathrm{Re}(s)>d(P)$. When $0<d(P)<\infty$, by defining another norm function by $N^1(p)=N(p)^{d(P)}$, we can normalize (P, N) to (P, N^1) which satisfies $d(P)=1$.

Example 1. Let A be a commutative finitely generated $\mathbf{Z}$-algebra, where $\mathbf{Z}$ denotes the ring of rational integers. Let $M(A)$ be the category of A-modules, and let $P=P(M(A))=P(A)$ be the "set" of all isomorphism classes of simple objects of $M(A)$. In this case P is actually a set and is consisting of isomorphism classes of simple A-modules. For each $p \in P$, let $N(p)=\sharp p$ be the cardinality of p as a set. (Each p is a finite set.) Then P is a prime set with the (integer valued) norm function N, and $d(P)$ is equal to the Krull dimension $\dim(A)$ of A. In particular, when $A=\mathbf{Z}$, $\zeta(s, P(\mathbf{Ab}))$ is equal to the Riemann zeta function $\zeta(s)$, where $\mathbf{Ab}=M(\mathbf{Z})$ is the category of abelian groups. (Note that $P(\mathbf{Ab})$ is the set of isomorphism classes of simple abelian groups, and that a simple abelian group is a finite cyclic group of prime order.) In other words, the Riemann zeta function is the zeta function of the category $\mathbf{Ab}$. In general, we expect that:

$$Z(s, P)=\zeta(s, P)\Gamma(s, P)=\prod_{m=0}^{2d(P)} Z_m(s, P)^{(-1)^{m+1}}$$

with the gamma factor $\Gamma(s, P)$, where $Z_m(s, P)$ is holomorphic on $\mathbf{C}$ having the functional equation for $s \to m-s$ with all zeros on $\mathrm{Re}(s)=m/2$. When $\zeta(s, P)$ is meromorphic on $\mathbf{C}$, we have an "explicit formula" attached to $\zeta(s, P)$ in the form $\sum_p M(p)=\sum_\lambda W(\lambda)$, where λ