85. *Extended Epstein's Zeta Functions over CM-fields*[*)]

By Hirofumi ISHIKAWA
Department of Mathematics, College of Arts and Sciences,
Okayama University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1984)

1. Introduction and statement of the results. The purpose of this note is to establish a relation between a series which derives from totally positive definite binary quadratic forms of discriminant Δ over a totally real algebraic number field F and Dedekind's Zeta function of CM-field $F(\sqrt{\Delta})$. In the case of Q, it has been done in [6, §4].

Let F be a totally real algebraic number field of degree n, $\mathfrak{o}_F$ the ring of integers in F, U_F the unit group of $\mathfrak{o}_F$ and $\Gamma=PSL_2(\mathfrak{o}_F)$. We assume the class number of F will be one in narrow sense. For any totally negative element Δ in $\mathfrak{o}_F$, denote by K the totally imaginary quadratic extention $F(\sqrt{\Delta})$ over F. Let Φ be the set of totally positive definite binary quadratic forms of discriminant Δ with $\mathfrak{o}_F$-coefficients. We consider Γ operates on Φ by

$$^{\sigma}\phi(x, y)=\phi(\alpha x+\gamma y,\ \beta x+\delta y), \qquad \left(\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right).$$

We define

$$(1)\qquad \zeta(s, \Delta)=\sum_{\phi\in\Phi/\Gamma}\ \sum_{(\mu,\nu)\in X/\mathrm{Aut}(\phi)} N_F(\phi(\nu, -\mu))^{-s} \qquad (\mathrm{Re}\,(s)>1).$$

Here, $X=\{\mathfrak{o}_F\times\mathfrak{o}_F-(0, 0)\}/U_F$, $\mathrm{Aut}\,(\phi)=\{\sigma\in\Gamma\,;\,{}^{\sigma}\phi=\phi\}$. Then $\zeta(s, \Delta)$ converges absolutely if $\mathrm{Re}\,(s)>1$, and uniformly if $\mathrm{Re}\,(s)\geqq 1+\varepsilon$ $(\varepsilon>0)$. So $\zeta(s, \Delta)$ is a holomorphic function in that region. It has been known from [3], [6] that $\zeta(s, \Delta)$ can be continued meromorphically to the whole plane and has a simple pole at $s=1$ because the first summation of (1) is a finite sum. We denote by D the discriminant of K over F, and by Δ_0 a totally negative integer such that $(\Delta_0)=D$. For a prime ideal $\mathfrak{p}$, put $\alpha_{\mathfrak{p}}=(1/2)(\mathrm{ord}_{\mathfrak{p}}(\Delta)-\mathrm{ord}_{\mathfrak{p}}D)$ and $\nu_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}D$. For an even prime ideal $\mathfrak{p}$, let $e_{\mathfrak{p}}$ be the ramification index of $\mathfrak{p}$ in F. If $\mathfrak{p}$ ramifies in K, we define a non-negative integer $k_{\mathfrak{p}}$ by

$$\max\{0\leqq k_{\mathfrak{p}}\leqq(\nu_{\mathfrak{p}}/2)+1\,;\ x^2\equiv\Delta_0 \bmod \mathfrak{p}^{2e_{\mathfrak{p}}+2k_{\mathfrak{p}}} \text{ is solvable for } x\in\mathfrak{o}_F\},$$

otherwise, we put $k_{\mathfrak{p}}=0$. We say Δ is exceptional if $k_{\mathfrak{p}}\geqq 1$.

Theorem. *For a non-exceptional Δ, if $\alpha_{\mathfrak{p}}\geqq 0$ for all $\mathfrak{p}$, we have*

$$(2)\qquad \zeta(s, \Delta)=\zeta_K(s)\sum_{\mathfrak{n}\mid\mathfrak{f}}\mu(\mathfrak{n})\chi_{\Delta}(\mathfrak{n})N_F(\mathfrak{n})^{-s}\sigma_{1-2s}(\mathfrak{f}/\mathfrak{n}),$$

*) This work was started by the author while visiting at The University of Washington, Seattle, U.S.A.