9. On Generalized Gelfand Pairs

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1. Introduction. Let G be a unimodular, locally compact second countable (lcsc) group and H a closed unimodular subgroup. Put $S=G/H$ and denote by $D(S)$ the space of Schwartz-Bruhat functions on S, endowed with the usual inductive-limit topology. Let $D'(S)$ be the topological anti-dual of $D(S)$, endowed with the strong topology. Let π be a continuous irreducible unitary representation of G on the Hilbert space $\mathcal{H}$.

We consider the following three problems:

(i) Can π be realized on a Hilbert-subspace of $D'(S)$: does there exist a continuous linear injection $j:\mathcal{H}\to D'(S)$ such that

$$j\pi(x)=L_x j$$

for all $x\in G$ (L_x denotes left translation by x)?

(ii) What can be said about uniqueness (up to scalar multiplication) of such a realization?

(iii) In case of unique realization, does $j(\mathcal{H})$ consist of functions (C^∞, locally L^2)?

Some results on these problems, obtained by L. Schwartz, P. Cartier, J. Faraut, G. E. F. Thomas, F. J. M. Klamer together with the author's results will be reported in this work. Details of the proofs of the author's results will be published elsewhere.

2. Realization on a Hilbert-subspace. We start with a result which can be obtained from [2] and [3]. Let G be a lcsc group, H a closed subgroup of G. For technical reasons, we assume both G and H to be unimodular. Put $S=G/H$. Let π be an irreducible unitary representation of G on the Hilbert space $\mathcal{H}$. The subspace $\mathcal{H}_\infty$ of C^∞-vectors in $\mathcal{H}$ can be endowed with a natural Sobolev-type topology (cf. [2], §1). $\mathcal{H}_\infty$ is G-invariant. Denote $\mathcal{H}_{-\infty}$ the anti-dual of $\mathcal{H}_\infty$. Then $\mathcal{H}_\infty\subset\mathcal{H}\subset\mathcal{H}_{-\infty}$ and the injections are continuous. G acts on $\mathcal{H}_{-\infty}$, the corresponding representation is called $\pi_{-\infty}$. Define

$$\mathcal{H}^H_{-\infty}=\{a\in\mathcal{H}_{-\infty}\mid\pi_{-\infty}(h)a=a \text{ for all } h\in H\}.$$

Then we have the following result.

Theorem 1. *π can be realized on a Hilbert-subspace of $D'(S)$ if and only if $\mathcal{H}^H_{-\infty}\neq(0)$. There is a one-to-one correspondence between the non-zero elements of $\mathcal{H}^H_{-\infty}$ and the continuous linear injections $j:\mathcal{H}\to D'(S)$ satisfying $j\pi(x)=L_x j(x\in G)$. To $a\neq 0$ in $\mathcal{H}^H_{-\infty}$ corresponds*