77. The Fabry-Ehrenpreis Gap Theorem for Hyperfunctions

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In [7], we have shown that the Fabry-type gap theorems can be most neatly handled by the aid of linear differential equations of infinite order, thus realizing an ideal of Ehrenpreis [3]. Although the classical gap theorems refer to holomorphic functions, it is evident that they are closely related to the analysis of Fourier series on a real domain. The relation is most obvious in the one-dimensional case:

Let $f_+(z)$ (resp., $f_-(z)$) denote $\sum_{n\geqq 0} c_n \exp(ia_n z)$ (resp., $\sum_{n<0} c_n \cdot \exp(ia_n z)$) ($c_n \in \boldsymbol{C}$, $a_n \in \boldsymbol{R}$ and $i=\sqrt{-1}$) and suppose that $f_+(z)$ (resp., $f_-(z)$) determines a holomorphic function on $\{z \in \boldsymbol{C};\ \mathrm{Im}\, z>0\}$ (resp., $\{z \in \boldsymbol{C};\ \mathrm{Im}\, z<0\}$). Suppose further that the sequence a_n is sufficiently lacunary so that Theorem 1 of [7] is applicable to them. Let $f(x)$ denote the hyperfunction determined by the pair of holomorphic functions $f_+(z)$ and $f_-(z)$, and suppose that $f(x)$ vanishes near $x=0$. This means, by the definition, that there exists a holomorphic function $F(z)$ defined on $\{z \in \boldsymbol{C};$ either $\mathrm{Im}\, z\neq 0$ or $|\mathrm{Re}\, z|<c\ (c>0)\}$ which coincides with $f_\pm(z)$ on $\{z \in \boldsymbol{C};\ \pm\mathrm{Im}\, z>0\}$, respectively. Then the gap theorem for holomorphic functions entails that both $f_+(z)$ and $f_-(z)$ are holomorphic in a neighborhood of the real axis $\boldsymbol{R}$, and hence their difference $f(x)$ is analytic on $\boldsymbol{R}$. Since $f(x)$ vanishes near $x=0$, this implies that $f(x)$ is identically zero.

In the higher dimensional case, however, such a straightforward connection cannot be observed immediately because of the complexity of the notion of the vanishing of a hyperfunction; it requires a cohomological language. (See [4], Chap. 1, §2, for example.) Still, this trouble due to the higher dimensionality of the problem is only a technical matter, as is usually the case in dealing with hyperfunctions; we can obtain the same result also for the higher dimensional case. This is what we want to report here.

In what follows, for a sequence $a(l)$ ($l \in \boldsymbol{N}=\{0, 1, 2, \cdots\}$) of m-dimensional real vectors, we let $a_j(n)$ ($j=1, \cdots, m$; $n \in \boldsymbol{N}$) denote its j-th reduced sequence in the sense of [7], Definition 1. We also denote $\sum_{j=1}^m |a(l)_j|$ by $|a(l)|$, where $a(l)_j$ denotes the j-th component of $a(l)$.

Theorem. *Let $a(l)$ ($l \in \boldsymbol{N}$) be a sequence of m-dimensional real vectors such that its j-th reduced sequence $a_j(n)$ satisfies the following*