76. On Totally Multiplicative Signatures of Natural Numbers

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1. Introduction. Let $\mathbf{N}$ be the set of all natural numbers and σ a mapping from $\mathbf{N}$ to the set $\{\pm 1\}$ satisfying the condition $\sigma(ab)=\sigma(a)\sigma(b)$ for all $a, b \in \mathbf{N}$. We call such a mapping σ *a totally multiplicative signature*. We have $\sigma(a^2)=1$, particularly $\sigma(1)=1$. The constant signature $\sigma(a)=1$ for all $a \in \mathbf{N}$ is called *trivial*. In the following, we are concerned with non-trivial totally multiplicative signatures, called simply signatures and denoted by σ. Let $\Pi(\sigma)$ be the set of all primes p, for which $\sigma(p)=-1$. σ is obviously determined by $\Pi(\sigma)$. When $\Pi(\sigma)$ coincides with the set of all primes, then σ is Liouville's function λ. S. Chowla conjectured that, given any finite sequence $\varepsilon_1, \cdots, \varepsilon_g$, $\varepsilon_m=\pm 1$, then $\lambda(x+m)=\varepsilon_m (1 \leqslant m \leqslant g)$ will have infinitely many solutions (cf. [1], [5]). In [4], I. Schur and G. Schur proved that the followings are the only signatures for which $\sigma(x)=\sigma(x+1)=\sigma(x+2)=1$ does not occur.

I. If $\sigma(3)=1$, then $\sigma(3n+1)=1$, $\sigma(3n+2)=-1$, $\sigma(3^k t)=\sigma(t)$ for all n, k, t with $(t, 3)=1$.

II. If $\sigma(3)=-1$, then $\sigma(3n+1)=1$, $\sigma(3n+2)=-1$, $\sigma(3^k t)=(-1)^k \sigma(t)$ for all n, k, t with $(t, 3)=1$.

Furthermore they proved that $\sigma(x)=1$, $\sigma(x+1)=-1$, $\sigma(x+2)=1$ has always a solution for any σ.

In this paper we prove the following theorem.

Theorem. *Let σ be a totally multiplicative signature for which $\Pi(\sigma)$ contains at least two primes. Then*

(i) $\sigma(x)=-1$, $\sigma(x+1)=-1$ *has infinitely many solutions,*

(ii) $\sigma(x)=-1$, $\sigma(x+1)=1$, $\sigma(x+2)=-1$ *has a solution and if $\sigma(2)=1$, it has infinitely many solutions.*

Our result contains a special case of Chowla's conjecture.

Henceforth we simply write either $(n)_+$ or $(n)_-$ instead of $\sigma(n)=1$ or $\sigma(n)=-1$, respectively.

2. Proof of Theorem. Let p, q be the smallest and the next smallest elements of $\Pi(\sigma)$. Then we have $1<p<q$, $(p, q)=1$.

Proof of (i). The congruence $qx \equiv 1 \pmod p$ has a unique solution x_0 in the interval $1 \leqslant x \leqslant p-1$. So there exists $r \in \mathbf{N}$ such that $qx_0=pr+1$. Similarly the congruence $qy \equiv -1 \pmod p$ has a unique