58. Variational Problems Governed by a Multi-Valued Differential Equation

By Toru MARUYAMA
Department of Economics, Keio University

(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1984)

1. Introduction. Throughout this paper, $\mathfrak{H}$ stands for a real Hilbert space of finite dimension and let a correspondence (=multi-valued mapping) $\Gamma:[0,T]\times\mathfrak{H}\twoheadrightarrow\mathfrak{H}$ be given. Define $\Delta(a)$ as a set of all elements x of $\mathcal{W}^{1,2}([0,T],\mathfrak{H})$ that satisfy

$$\begin{cases}\dot{x}(t)\in\Gamma(t,x(t)) & \text{a.e.}\\ x(0)=a\end{cases}\qquad (*)$$

In the previous paper [1], we proved, under some assumptions, that $\Delta(a)$ is non-empty and it depends continuously, in some sense, upon the initial value a. In the present paper, we shall examine a couple of variational problems governed by a multi-valued differential equation of the form (*), and establish sufficient conditions which assure the existence of optimal solutions for them.

2. Review of the Previous Result. For the sake of the readers' convenience, we shall summarize the main result obtained in Maruyama [1].

Assumption 1. Γ is compact-convex-valued; i.e. $\Gamma(t,x)$ is a non-empty, compact and convex subset of $\mathfrak{H}$ for all $t\in[0,T]$ and all $x\in\mathfrak{H}$.

Assumption 2. The correspondence $x\twoheadrightarrow\Gamma(t,x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t\in\Gamma[0,T]$.

Assumption 3. The correspondence $t\twoheadrightarrow\Gamma(t,x)$ is measurable for each fixed $x\in\mathfrak{H}$.

For the concept of "measurability" of a correspondence, see Maruyama [6] Chap. 6.

Assumption 4. There exists $\psi\in L^2([0,T],\boldsymbol{R}_+)$ such that

$$\Gamma(t,x)\subset S_{\psi(t)}\qquad\text{for every } (t,x)\in[0,T]\times\mathfrak{H},$$

where $S_{\psi(t)}$ is the closed ball in $\mathfrak{H}$ with the center 0 and the radius $\psi(t)$.

Existence Theorem (Maruyama [1]). *Suppose that Γ satisfies Assumption* 1-4, *and let A be a non-empty, convex and compact subset of $\mathfrak{H}$. Then*

(i) *$\Delta(a^*)\neq\phi$ for any $a^*\in A$, and*

(ii) *the correspondence $\Delta:A\twoheadrightarrow\mathcal{W}^{1,2}$ is compact-valued and upper hemi-continuous (abbreviated as u.h.c.) on A in the weak topology for $\mathcal{W}^{1,2}$.*

3. Variational Problem (1). Let $u:[0,T]\times\mathfrak{H}\times\mathfrak{H}\to\boldsymbol{R}_+$ be a given