# 6. The Steffensen Iteration Method for Systems of Nonlinear Equations

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**1. Introduction.** Let $x=(x_1, x_2, \cdots, x_n)$ be a vector in $R^n$ and $D$ a region contained in $R^n$. Let $f_i(x)(1\leq i\leq n)$ be real-valued nonlinear functions defined on $D$ and $f(x)=(f_1(x), f_2(x), \cdots, f_n(x))$ an $n$-dimensional vector-valued function. Then we shall consider a system of nonlinear equations

$$x=f(x), \tag{1.1}$$

whose solution is $\bar{x}$. Denote by $\|x\|$ and $\|A\|$ the $l_\infty$-norm and the corresponding matrix norm, respectively. That is,

$$\|x\|=\max_{1\leq i\leq n}|x_i| \quad \text{and} \quad \|A\|=\max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|,$$

where $A=(a_{ij})$ is an $n\times n$ matrix.

In generalizing the Aitken $\delta^2$-process in one dimension to the case of $n$-dimensions, Henrici [1, p. 116] has considered the following formula, which is called the Aitken-Steffensen formula:

$$y^{(k)}=x^{(k)}-\Delta X^{(k)}(\Delta^2 X^{(k)})^{-1}\Delta x^{(k)}. \tag{1.2}$$

Furthermore, he has conjectured the following: We may hope that $y^{(k)}$ defined by (1.2) is closer to $\bar{x}$ than $x^{(k)}$, provided that the matrices $\Delta X^{(k)}$ and $\Delta^2 X^{(k)}$ are invertible. But he has not given mathematical certification to such a conjecture.

In [2], we have studied the above Aitken-Steffensen formula and shown [2, Theorem 2].

The purpose of this paper is to show Theorem 1 by considering a method of iteration, often called the Steffensen iteration method. Theorem 1 is an improvement on the result of [2, Theorem 2].

**2. Statement of results.** Define $f^{(i)}(x)\in R^n$ $(i=0, 1, 2, \cdots)$ by

$$f^{(0)}(x)=x,$$
$$f^{(i)}(x)=f(f^{(i-1)}(x)) \qquad (i=1, 2, \cdots).$$

Put

$$d^{(0,k)}=x^{(k)}-\bar{x},$$
$$d^{(i,k)}=f^{(i)}(x^{(k)})-\bar{x} \qquad \text{for } i=1, 2, \cdots.$$

Then an $n\times n$ matrix $D(x^{(k)})$ is defined as

$$D(x^{(k)})=(d^{(0,k)}, d^{(1,k)}, \cdots, d^{(n-1,k)}).$$

Throughout this paper, we shall assume the following five con-