44. On a Multi-valued Differential Equation: An Existence Theorem

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(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1984)

1. Introduction. Let $\mathfrak{H}$ be a real Hilbert space of finite dimension and a correspondence (=multi-valued mapping) $\Gamma: [0, T]\times\mathfrak{H} \twoheadrightarrow \mathfrak{H}$ is assumed to be given. The compact interval $[0, T]$ is endowed with the usual Lebesgue measure dt. In this paper, we shall establish a sufficient condition which assures the existence of solutions of a multi-valued differential equation of the form:

$$\begin{cases} \dot{x}(t) \in \Gamma(t, x(t)) \\ x(0)=a, \end{cases} \qquad (*)$$

where a is a fixed vector in $\mathfrak{H}$. The author is very much indebted to the works of Attouch-Damlamian [2] and Castaing [3], which treat the closely related problems. The purpose of this paper is to reconstruct their theories in the framework of the Sobolev space $\mathcal{W}^{1,2}$. We shall then proceed, in another paper [7], to examining some variational problems governed by a multi-valued differential equation like (*). A couple of existence theorems of optimal solutions for them will be shown there.

2. Assumption. Let us begin by specifying some assumptions imposed on the correspondence Γ.

Assumption 1. Γ is compact-convex-valued; i.e. $\Gamma(t, x)$ is a non-empty, compact and convex subset of $\mathfrak{H}$ for all $t\in[0, T]$ and all $x\in\mathfrak{H}$.

Assumption 2. The correspondence $x \twoheadrightarrow \Gamma(t, x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t\in[0, T]$.

Assumption 3. The correspondence $t \twoheadrightarrow \Gamma(t, x)$ is measurable for each fixed $x\in\mathfrak{H}$. For the concept of "measurability" of a correspondence, see [6] Chap. 6.

Assumption 4. There exists $\psi\in L^2([0, T], \boldsymbol{R}_+)$ such that $\Gamma(t, x) \subset S_{\psi(t)}$ for every $(t, x)\in[0, T]\times\mathfrak{H}$, where $S_{\psi(t)}$ is the closed ball in $\mathfrak{H}$ with the center 0 and the radius $\psi(t)$.

3. Preliminary Lemmas.

Lemma 1 (Castaing [3]). *Let Γ be a correspondence which satisfies Assumption* 1-3. *And let $x: [0, T]\to\mathfrak{H}$ be any measurable mapping. Then there exists a closed-valued measurable correspondence $\Sigma: [0, T] \twoheadrightarrow \mathfrak{H}$ such that $\Sigma(t)\subset\Gamma(t, x(t))$ for all $t\in[0, T]$.*

Lemma 2. *Let A be a non-empty, convex and compact subset of*