40. On the Euler-Poisson-Darboux Equation and the Toda Equation. I

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§1. **Summary.** The Toda equation with two time variables

(1.1) $$XY\log t_n = t_{n+1}t_{n-1}/t_n^2 \qquad \left(X=\frac{\partial}{\partial x},\ Y=\frac{\partial}{\partial y},\ t_n=t_n(x,y)\right)$$

can be solved using solutions of the Euler-Poisson-Darboux equations ([1])

(1.2) $$(XY+(\alpha+\beta-1-2n)\varphi^{-1}X-(n-\alpha)(n-\beta)\varphi^{-2})u_n=0,$$

where $\varphi=x-y$. Rational solutions, Gauss hypergeometric function solutions and solutions which can be expressed by hypergeometric functions with two variables (Appell hypergeometric functions F_1, F_2 and F_3 are included) are obtained. K. Okamoto ([2]) also found these hypergeometric solutions.

§2. **Bäcklund transformation.** When t_n satisfies (1.1)

(2.1) $$r_n=XY\log t_n, \qquad s_n=Y\log t_{n-1}/t_n$$

satisfies

(2.2) $$Yr_n=r_n(s_n-s_{n+1}), \qquad Xs_n=r_{n-1}-r_n.$$

Eliminating s_n we have

(2.3) $$XY\log r_n=r_{n+1}-2r_n+r_{n-1}.$$

This form of the Toda equation was found by G. Darboux ([1]). As was shown in our previous work ([3])

(2.4) $$t_n=F(n)\varphi^{-f(n)},$$

where $f(n)=(n-\alpha)(n-\beta)$, α and β are arbitrary constants,

$$F(n+1)F(n-1)/F(n)^2=-f(n), \qquad F(0)=F(1)=1,$$

satisfies the Toda equation (1.1). Corresponding

(2.5) $$r_n=-f(n)\varphi^{-2}, \qquad s_n=(\alpha+\beta+1-2n)\varphi^{-1}$$

satisfies the Toda equation (2.2). This simple important solution r_n was first found by G. Darboux ([1]). For these special solutions we put

(2.6) $$M_n=XY+(\alpha+\beta-1-2n)\varphi^{-1}X-(n-\alpha)(n-\beta)\varphi^{-2},$$
$$X_n=((n-\alpha)(n-\beta))^{-1}\varphi^2X, \qquad Y_n=Y+(\alpha+\beta-1-2n)\varphi^{-1}.$$

Define

(2.7) $$T=\{u_n\,;\,M_0u_0=0,\ u_{n+1}=Y_nu_n\ (n\geqslant 0),\ u_{n-1}=X_nu_n\ (n\leqslant 0)\}.$$

Theorem 2.1 (Bäcklund transformation). *If* $u_n\in T$ *then we have* $M_nu_n=0,\ \ u_{n+1}=Y_nu_n,\ \ u_{n-1}=X_nu_n\ \ (n=0,\pm 1,\pm 2,\cdots)$ *and* $\tau_n=u_nt_n$