37. On Marot Rings

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§1. **Introduction.** Throughout the paper, a ring means a commutative ring with identity. A non-zerodivisor of a ring is said to be regular, and an ideal containing regular elements is said to be regular. A ring R is said to be a *Marot ring* (cf. [3]), if each regular ideal of R is generated by regular elements. The main purpose of this paper is to solve the following question on Marot rings posed by Portelli-Spangher [6]. : Let $\mathfrak{a}$ be an ideal of a ring R. We denote the set of regular elements contained in $\mathfrak{a}$ by $\mathrm{Reg}(\mathfrak{a})$. We say that a ring R has *property* (FU), if $\mathrm{Reg}(\mathfrak{a})\subset\bigcup_{i=1}^n\mathfrak{a}_i$ implies $\mathfrak{a}\subset\bigcup_{i=1}^n\mathfrak{a}_i$ for each family of a finite number of regular ideals $\mathfrak{a},\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_n$. If R has property (FU), then R is a Marot ring. The question is: *Does a Marot ring have property* (FU)?

§2. **Answer to the question.** Let us begin by some lemmas.

Lemma 1. *Let R be a ring.*

(1) *R is a Marot ring if and only if an ideal (r,s) is generated by regular elements for each regular element r of R and for each element $s\in R$.*

(2) *R has property (FU) if and only if* $\mathrm{Reg}((r,s))\subset\bigcup_{i=1}^n\mathfrak{a}_i$ *implies $(r,s)\subset\bigcup_{i=1}^n\mathfrak{a}_i$ for each pair of elements r,s of R with r regular and for each family of a finite number of regular ideals $\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_n$.*

Let A be a ring, and let M be an A-module. We construct a semidirect product R by the principle of idealization ([5, Chap. 1, n°1]). That is, $R=A\oplus M$ and for elements $f+x$ and $g+y$ of R we set $(f+x)(g+y)=fg+(fy+gx)$, where $f,g\in A$ and $x,y\in M$.

Lemma 2. *Let $f+x$ be an element of R. Then $f+x$ is a regular element of R if and only if f is a regular element of R.*

Let p be a prime number, and let k be a finite field of characteristic p. We denote by A the subring $k[X^p,X^{p+1},X^{p+2},\cdots]$ of the polynomial ring $k[X]$. Let $\{F_0,F_1,\cdots,F_n,G_1,G_2,\cdots\}$ be a set of irreducible polynomials of $k[X]$ such that (1) $F_0=X$ and $F_1=1+X$, (2) $\deg(F_i)<2p$ for each i, (3) $\deg(G_j)\geq 2p$ for each j, (4) any two elements of the set are not associated and (5) each irreducible polynomial of $k[X]$ is associated with some element of the set. We denote $k[X]/(G_j)$ by K_j. K_j is naturally an A-module. We construct a direct sum M of A-modules $K_1,K_2,K_3,\cdots$, and construct a semidirect product $R=A$