31. A Varifold Solution of the Nonlinear Wave Equation of a Membrane

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§1. **Introduction.** Let U be a bounded domain in $\mathbf{R}^n$ with the boundary ∂U which is a Lipschitz manifold. Let $D_j=\partial/\partial x_j$, $j=1, 2, \cdots, n$, and $D_t=\partial/\partial t$. Then the nonlinear wave equation we shall consider is as follows:

(1) $$D_t^2 u(t, x)-\sum_{j=1}^n D_j\{D_j u(t, x)(1+|Du(t, x)|^2)^{-1/2}\}=0.$$

(2) $$u(t, x)=u_0(x), \qquad D_t u(0, x)=u_1(x).$$

(3) $$u(t, x)=g(x) \qquad \text{for } x \text{ in } \partial U.$$

The global existence of a weak solution of the above equation is not yet proved in general. (See §2 below for the definition of the weak solution.) In this paper, we shall try to treat the equations (1)–(3) by virtue of the theory of varifolds (cf. [1] and [2]) and prove the global existence of a varifold solution of them. Although a varifold solution is quite a weak notion, the varifold solution existence of which we can prove satisfies a generalized energy conservation law and is a solution of a problem of calculus of variation, which is a natural generalization of Hamilton's principle:

(4) $$\delta\int_0^T dt\int_U \left\{\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2-\sqrt{1+|Du|^2}\right\}dx=0.$$

Proofs of the results in this paper will be published elsewhere.

§2. **A weak solution.** We shall denote by $BV(U)$ the space of all functions of bounded variation in U, that is, $u\in BV(U)$ if and only if $u\in L^1(U)$ and its gradient $Du=(D_1u, D_2u, \cdots, D_nu)$ is a vector valued Radon measure (cf. [3]). We denote its total variation by $|Du|$. The Sobolev space $H^1(U)$ of order 1 is contained in $BV(U)$. If $u\in BV(U)$ then its trace γu from the interior of U is a function in $L^1(\partial U)$. For u in $BV(U)$, the set $E_u=\{(x, y)\in U\times\mathbf{R} \mid y<u(x)\}$ is a Caccioppoli subset of $\mathbf{R}^{n+1}$. At each point (x, y) of the reduced boundary $\partial^* E_u$ of E_u, we can define the exterior unit normal $\nu(x, y)=(\nu_1(x, y), \nu_2(x, y), \cdots, \nu_n(x, y), \nu_{n+1}(x, y))$ to E_u. The characteristic function χ_E of E_u is of bounded variation. $|D\chi_E|$ denotes the total variation of the gradient $D\chi_E$.

Definition 2.1. Assume that $u_0\in H^1(U)$, $u_1\in L^2(U)$ and that g is the trace of some function in $BV(U)$. Then a function $u(t, x)\in L^1_{\mathrm{loc}}(\mathbf{R}\times U)$ is a weak solution of the equations (1), (2) and (3) if the following