29. On an Identity of Desboves

By Jasbir Singh CHAHAL
Mathematical Department, Brigham Young University

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§1. **Introduction.** A. Desboves (cf. [2], see also [3], p. 631, line 2) employed the identity

$$(1)\qquad \begin{aligned}&(y^2+2xy-x^2)^4+(2x^3y+x^2y^2)(2x+2y)^4\\&=(x^4+y^4+10x^2y^2+4xy^3+12x^3y)^2\end{aligned}$$

to show, among others, that $x^4+ay^4=z^2$ is solvable in $\boldsymbol{Z}$ if a is of the form $(2x+y)x^2y$ or $2x^2+y^4$. The purpose of this note is to show that this identity can also be used to get a point of infinite order of $E(\boldsymbol{Q})$, the group of rational points on certain elliptic curves E of the form

$$(2)\qquad y^2=x^3+Ax, \qquad A\in\boldsymbol{Q}.$$

Here, without loss of generality, we can assume that A is a non-zero integer, free of fourth powers. In another context it has been widely conjectured (cf. [5] or the table on p. 147 of [4]) that if a positive integer $n\equiv 5, 6, 7$ (mod. 8) then n is a congruent number, i.e., it is the area of a right triangle of all sides rational. We shall rather show that any residue class modulo 8 contains infinitely many congruent numbers.

§2. **The main result.** First we state the following theorem which we shall need in the sequel and which was proved independently by E. Lutz and T. Nagell (cf. [1], p. 264, Theorem 22.1).

Theorem 1. *Suppose $P=(x, y)\in E(\boldsymbol{Q})$ is a point of finite order on the elliptic curve $y^2=x^3+Ax+B$ with $A, B\in\boldsymbol{Z}$. Then x and y are necessarily integers.*

Theorem 2. *For any integer $\lambda\neq 0$, let E_λ be the curve*

$$(3)\qquad y^2=x^3+A_\lambda x,$$

where $A_\lambda=8\lambda(2\lambda-1)^2$. Then $E_\lambda(\boldsymbol{Q})$ has a point of infinite order.

Proof. A solution (s, t, u) with $t\neq 0$ of $s^4+At^4=u^2$ leads to a solution $x=s^2/t^2$ and $y=su/t^3$ of (2). The following identity

$$\begin{aligned}&(1-12\lambda+4\lambda^2)^4+8\lambda(2\lambda-1)^2(2(1+2\lambda))^4\\&=(1+40\lambda-104\lambda^2+160\lambda^3+16\lambda^4)^2,\end{aligned}$$

which follows from (1) by putting $x=1-2\lambda$, $y=4\lambda$ gives a rational point $P=(x, y)$ on (3) with

$$x=x(\lambda)=\frac{(1-12\lambda+4\lambda^2)^2}{4(1+2\lambda)^2},$$

$$y=y(\lambda)=\frac{(1-12\lambda+4\lambda^2)(1+40\lambda-104\lambda^2+160\lambda^3+16\lambda^4)}{8(1+2\lambda)^3}$$