27. On the T-Genus of Knot Cobordism

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The second and third authors introduced in [4] an integral invariant $T(k)$ of a classical (tame) knot k such that (1) $T(k)$ is invariant under knot cobordism, (2) $g^*(k) \leq T(k)$ and (3) $T(k) \equiv \mathrm{Arf}\,(k)$ (mod 2), where $g^*(k)$ and Arf (k) are the slice genus and the Arf invariant of k, respectively. We call $T(k)$ the *T-genus* of k. The purpose of this paper is to give an alternative definition of the T-genus and to note that the T-genus induces a metric function d_T on the knot cobordism group X defined by Fox-Milnor in [2]. Some properties of the space (X, d_T) are described without proof here, but more properties containing the details will appear in "On a geometry of the knot cobordism group".

Let R be the Borromean rings (cf. Fox [1, p. 131]). We denote by $k\#_b(R_1+\cdots+R_r)$ a knot obtained by a fusion from the split union $k+R_1+\cdots+R_r$ of a knot k and r copies $R_1, \cdots, R_r$ of R (see [3] for the definition of fusion). Note that the knot type of the resulting knot depends on a choice of the fusion-bands.

Lemma 1. *Given a knot k with $T(k) \geq 1$, there is a knot $k' = k\#_b R$ such that $T(k') \leq T(k) - 1$.*

Proof. Let $T(k)=r$. By [4, Proof of Theorem (2)] there is a cobordism surface of genus 0 between k and $R_1+\cdots+R_r$. Then we obtain a cobordism surface of genus 0 between $k+R_1$ and $R_2+\cdots+R_r$. By the deformation theory [3] of cobordism surface, some $k' = k\#_b R_1$ is cobordant to some $k'' = 0\#_b(R_2+\cdots+R_r)$ (0 is the trivial knot). Since $T(k') = T(k'')$ and $T(k'') \leq r-1$, the desired result follows.

For a knot k the minimal number of r such that some $k\#_b(R_1+\cdots+R_r)$ is a slice knot is denoted by $B(k)$.

Theorem 2. $T(k) = B(k)$.

Proof. By Lemma 1 $T(k) \geq B(k)$, since $(\cdots((k\#_b R_1)\#_b R_2)\cdots)\#_b R_r$ is modified as $k\#_b(R_1+\cdots+R_r)$ by deforming and sliding the fusion-bands (cf. [3, Lemma 1.14]). To see that $T(k) \leq B(k)$, let $B(k)=s$. Since some $k\#_b(R_1+\cdots+R_s)$ is slice, $k+R_1+\cdots+R_s$ bounds a surface of genus 0 in $R^3[0, +\infty)$. So there is a cobordism surface of genus 0 between k and $R_1+\cdots+R_s$. By the deformation theory [3], k is cobordant to some $k' = 0\#_b(R_1+\cdots+R_s)$. Then $T(k) = T(k') \leq s = B(k)$, completing the proof.