20. On Certain Cubic Fields. I

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1. We shall use the following notations: For an algebraic number field F, the ring of integers, the group of units, the group of units with norm 1 and the discriminant of F by $\mathcal{O}_F, E_F, E_F^+$, and D_F respectively. The discriminant of an algebraic number θ will be denoted by $D(\theta)$ and the discriminant of a polynomial $f(x) \in \boldsymbol{Z}[x]$ by D_f.

Now let $K/\boldsymbol{Q}$ be totally real and cubic. For $\alpha \in K$, α', α'' will denote the conjugates of α. We define after [3] the function S from $K^\times$ to $\boldsymbol{R}$ by

$$S(\alpha)=\frac{1}{2}\{(\alpha-\alpha')^2+(\alpha'-\alpha'')^2+(\alpha''-\alpha)^2\}.$$

Let $1, \xi, \eta$ be a $\boldsymbol{Z}$ basis of $\mathcal{O}_K$. For $\alpha=x+y\xi+z\eta \in \mathcal{O}_K$, $x, y, z \in \boldsymbol{Z}$, $S(\alpha)$ is a positive definite quadratic form in y, z, so that $S(\alpha)$ has a minimal value on E_K.

Let us denote $\mathcal{A}(K)=\{\varepsilon \in E_K^+ \mid \varepsilon \neq 1, S(\varepsilon) \text{ is minimum}\}$ and $\mathcal{B}_{\varepsilon_1}(K) =(E_K^+ \backslash \{\varepsilon_1^n ; n \in \boldsymbol{Z}\}) \cap \mathcal{A}(K)$ for $\varepsilon_1 \in \mathcal{A}(K)$.

In [5], H. J. Godwin announced the following conjecture:

Conjecture. *If $\varepsilon_1 \in \mathcal{A}(K)$, $\varepsilon_2 \in \mathcal{B}_{\varepsilon_1}(K)$ and $S(\varepsilon_1)>9$, then $\varepsilon_1, \varepsilon_2$ generate E_K^+: $E_K^+=\langle\varepsilon_1, \varepsilon_2\rangle$.*

The purpose of this note is to show that this conjecture holds in certain cases. We shall prove:

Theorem. *Let $K=\boldsymbol{Q}(\theta)$, $\mathrm{Irr}\,(\theta : \boldsymbol{Q})=f(x)=x^3-mx^2-(m+3)x-1$, $m \in \boldsymbol{Z}$, with square free m^2+3m+9. Then we have $\theta \in \mathcal{A}(K)$, $-1-\theta \in \mathcal{B}_\theta(K)$ and $E_K^+=\langle\theta, -1-\theta\rangle$.*

Remark 1. It is easy to see that $f(x)$ is irreducible, so that $K/\boldsymbol{Q}$ is cubic. It is cyclic and consequently totally real, because $\sqrt{D_f} \in \boldsymbol{Z}$. It is also easy to see that we can limit our consideration to the case $m \geqq -1$. This will be supposed throughout in the sequel.

Remark 2. This kind of fields has been considered by K. Uchida [8], E. Thomas [7] and M.-N. Gras [4].

2. The following propositions will be utilized for the proof of Theorem.

Proposition 1 (H. Brunotte and F. Halter-Koch [1]). *Let $\varepsilon_1 \in \mathcal{A}(K)$, $\varepsilon_2 \in \mathcal{B}_{\varepsilon_1}(K)$, then $(E_K^+ : \langle\varepsilon_1, \varepsilon_2\rangle) \leqq 4$.*

Proposition 2 (E. H. Grossman [6], M. Watabe [9]). *Suppose $K/\boldsymbol{Q}$ to be totally real, $l \in \boldsymbol{Z}$, $l \geqq 2$, $\delta \in E_K$. Then the only possible*