# 17. *Formation of Singularities for Hamilton-Jacobi Equation. I*

By Mikio TSUJI
Department of Mathematics, Kyoto Sangyo University

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§ 1. **Introduction.** This note is concerned with the singularities of global solution of Hamilton-Jacobi equation in two space dimensions:

$$(1)\qquad \frac{\partial u}{\partial t}+f\left(\frac{\partial u}{\partial x}\right)=0 \qquad \text{in } \{t>0,\ x\in R^2\},$$

$$(2)\qquad u(0,x)=\varphi(x)\in C_0^\infty(R^2),$$

where $C_0^\infty(R^2)$ is a set of $C^\infty$-functions whose supports are compact. In this note we assume that $f$ is $C^\infty$ and uniformly convex. It's well known that, even for smooth initial data, the Cauchy problem (1) and (2) doesn't admit a smooth solution for all $t$. Therefore we consider a generalized solution of (1), (2) whose definition will be given in § 2. The existence of global generalized solutions is already established by many authors. (See [1] and its references.)

For a single conservation law in one space dimension, a solution satisfying the entropy condition is piecewise smooth for any smooth initial data in $\mathscr{S}=\{$rapidly decreasing functions$\}$ except for a subset of the first category ([3]–[5] and [8]). T. Debeneix [2] treated certain systems of conservation law which is essentially equivalent to Hamilton-Jacobi equation (1) in $R^n$ ($n\leqslant 4$), and generalized the results of [8] to this case by the same method as [8].

One of the classical methods for solving first order non-linear equations is the characteristic one. Its weak point is that it's the local theory. The reason is due to the fact that a smooth mapping can't uniquely have the inverse at a point where its jacobian vanishes, i.e., that its inverse becomes many-valued there. Therefore the solution takes also many values in a neighborhood of critical points of a mapping $H_t$ defined in § 3. The aim of this note is to show how to choose up the reasonable value of its many values so that the solution is one-valued and continuous.

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§ 2. **Generalized solutions.** Let's put $p=(p_1,p_2)\in R^2$, and write

$$f'(p)=\left(\frac{\partial f}{\partial p_1}(p),\ \frac{\partial f}{\partial p_2}(p)\right) \quad\text{and}\quad f''(p)=\left[\frac{\partial^2 f}{\partial p_i\partial p_j}(p)\right]_{1\leqslant i,\,j\leqslant 2}.$$