16. *Transmutation Theory for Certain Radial Operators*

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1. **Introduction.** For $Q_0u=(\Delta_Q u')'/\Delta_Q$ based on a radial Laplace-Beltrami operator we studied general transmutation theory for operators $\hat{Q}u=Q_0u+\hat{q}(x)u$ in [4] using various solutions of $\hat{Q}u=-k^2u$ as important ingredients. In the present work we consider operators $\tilde{Q}u=x^2Q_0u+x^2\{k^2-\tilde{q}(x)\}u$ (with corresponding eigenfunction equations $\tilde{Q}u=\lambda^2u$) and will concentrate on the case $\Delta_Q=x^2$ which arises in various scattering problems (cf. [1], [2], [6]–[9], [11], [12]) so a certain amount of guideline information is available (cf. also [5]-B transmutes P into $Q, B: P\to Q$, if $QBf=BPf$ for suitable f). We show here that with suitable modifications most of the constructions and techniques of the $\hat{Q}$ theory have a version in the $\tilde{Q}$ theory and we describe some of the basic transmutations and connection formulas.

2. **Basic constructions.** We take $\Delta_Q=x^2$ and set $\varphi=xu$ so $\tilde{Q}u=\lambda^2u$ is

$$x^2\varphi''+x^2\{k^2-\tilde{q}(x)\}\varphi=\lambda^2\varphi \tag{2.1}$$

($\tilde{q}$ real) and one writes $\lambda^2=\sigma(\sigma+1)=\nu^2-1/4$ (so $\sigma\sim l=$angular momentum and $\nu\sim l+1/2$). We denote by $\varphi(\nu,k,x)$ the "regular solution" of (2.1) ($\varphi\sim x^{\nu+1/2}$ as $x\to 0$) and by $f(\nu,\pm k,x)$ the "Jost solutions" (e.g. $f(\nu,-k,x)\sim e^{ikx}$ as $x\to\infty$) with the "Jost function" $f(\nu,-k)=W(f(\nu,-k,x),\varphi(\nu,k,x))$ ($W(f,g)=fg'-f'g$). Assume e.g. $\int_0^\infty x|\tilde{q}|dx<\infty$ and $\int_0^\infty x^2|\tilde{q}|dx<\infty$ as in [2] but we do not emphasize hypotheses on $\tilde{q}$ (cf. [1], [6], [9], [11], [12]); we want mainly $\varphi\sim\varphi_0$ and $f\sim f_0$ as e.g. $|\nu|\to\infty$, $\mathrm{Re}\,\nu>0$, where (corresponding to $\tilde{q}=0$)

$$\varphi_0(\nu,k,x)=2^\nu\Gamma(\nu+1)k^{-\nu}x^{1/2}J_\nu(kx)\,;\quad f_0=((1/2)\pi kx)^{1/2}e^{(1/2)i\pi(\nu+1/2)}H^1_\nu(kx)\,; \tag{2.2}$$

($f_0=f_0(\nu,-k,x)$) and

$$f_0(\nu,-k)=2^\nu(2/\pi)^{1/2}\Gamma(\nu+1)k^{-\nu+1/2}\exp\{(1/2)i\pi(\nu-1/2)\}.$$

We think of k as fixed here and one knows then that $f(\nu,-k,x)$ is entire in ν while $\varphi(\nu,k,x)$ and $f(\nu,-k)$ are analytic for $\mathrm{Re}\,\nu>0$ (the range of analyticity can be enlarged with suitable hypotheses on $\tilde{q}$). We follow formally now the procedure in [2] with some refinements and elaboration. Thus set $g(\nu,-k,r)=f(\nu,-k,r)/r$ and let Z denote the zeros ν_j (if any) of $f(\nu,-k)$ in $\mathrm{Re}\,\nu>0$ with