141. On Certain Integrals over Spheres

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1983)

§1. Statement of the formula (F). Let us begin with a well-known formula of the complete elliptic integral

$$(1.1)\qquad \frac{2}{\pi}K \overset{\text{def}}{=\!=} \frac{2}{\pi}\int_0^{\pi/2}(1-\lambda\sin^2\theta)^{-1/2}d\theta = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\lambda\right),$$

where $\lambda\in \boldsymbol{C}$, $|\lambda|<1$ and ${}_2F_1$ is the Gauss' hypergeometric series. If we pass to the cartesian coordinates $x=\cos\theta$, $y=\sin\theta$ of the plane $\boldsymbol{R}^2$, then (1.1) becomes

$$(1.2)\qquad \int_{S^1}(1-\lambda y^2)^{-1/2}d\omega = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\lambda\right)$$

where, in general, S^{n-1} denotes the unit sphere of $\boldsymbol{R}^n$ with the center at the origin and $d\omega$ is the volume element of S^{n-1} such that the volume of S^{n-1} is 1. Notice here that y^2 is a degenerate quadratic form on $\boldsymbol{R}^2$.

In this note, we shall give a generalization of (1.2). Namely, along with a partition $n=p+q$, p, $q>0$, of an integer n, consider the decomposition $\boldsymbol{R}^n=\boldsymbol{R}^p\oplus\boldsymbol{R}^q$ of the euclidean space $\boldsymbol{R}^n$. When $z=(x,y)\in\boldsymbol{R}^n$, $x\in\boldsymbol{R}^p$, $y\in\boldsymbol{R}^q$, we have $Nz=Nx+Ny$, where $Nx=x_1^2+\cdots+x_p^2$, etc. Let a, b be non-negative integers such that $c=a+b>0$. Our generalization of (1.2) is the following formula:

$$(\mathrm{F})\qquad \int_{S^{n-1}}(1-\lambda(Nx)^a(Ny)^b)^{-s}d\omega = {}_{c+1}F_c(s,\alpha;\beta;a^ab^bc^{-c}\lambda),$$

where $\lambda\in\boldsymbol{C}$, $|\lambda|<1$, $s\in\boldsymbol{C}$ and

$$\alpha=\left(\frac{p}{2a},\frac{p+2}{2a},\cdots,\frac{p+2(a-1)}{2a},\frac{q}{2b},\frac{q+2}{2b},\cdots,\frac{q+2(b-1)}{2b}\right),$$

$$\beta=\left(\frac{n}{2c},\frac{n+2}{2c},\cdots,\frac{n+2(c-1)}{2c}\right).$$

Needless to say, (1.2) is a special case of (F) where $n=2$, $p=q=1$, $a=0$, $b=c=1$ and $s=1/2$. We remind the reader the definition of the (generalized) hypergeometric series which appears on the right hand side of (F). First, for $a\in\boldsymbol{C}$, $k\in\boldsymbol{Z}$, $k\geqq 0$, we put, following Appell,

$$(a,k)=\begin{cases} a(a+1)\cdots(a+k-1), & k>0,\\ 1, & k=0.\end{cases}$$

Next, for integers μ, $\nu\geqq 0$, consider vectors $\alpha=(\alpha_1,\cdots,\alpha_\mu)\in\boldsymbol{C}^\mu$, $\beta=(\beta_1,\cdots,\beta_\nu)\in\boldsymbol{C}^\nu$. The hypergeometric series ${}_\mu F_\nu(\alpha;\beta;z)$, $z\in\boldsymbol{C}$, is then defined by