136. An Integro-Differential Operator and the Associated Semigroup of Operators

By Tadashi UENO

College of General Education, University of Tokyo

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1983)

1. As in [4], the generator of a certain type of semigroup is represented as an integro-differential operator.

Here, we solve a converse problem for an operator of this type, which is spatially homogeneous on R^N. Let

(1) $$Af(x)=A_Df(x)+A_If(x),\qquad x=(x_1,\cdots,x_N)\in R^N,$$
$$A_Df(x)=\sum_{|\alpha|\leqslant m} a_\alpha D_\alpha f(x),$$
$$A_If(x)=\int_{R^N\setminus\{0\}}\{f(y+x)-\rho(y)\sum_{|\alpha|<n}1/\alpha!\,D_\alpha f(x)y_\alpha\}\mu(dy).$$

a_α's are complex constants and $D_\alpha f(x)=\partial^{|\alpha|}f(x)/\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}$, $|\alpha|=\alpha_1+\cdots+\alpha_N$, $\alpha!=\alpha_1!\cdots\alpha_N!$ and $y_\alpha=y_1^{\alpha_1}\cdots y_N^{\alpha_N}$ for multi-index $\alpha=(\alpha_1,\cdots,\alpha_N)$. μ is a complex valued σ-finite measure on $(R^N\setminus\{0\},\mathscr{B}_{R^N\setminus\{0\}})$ such that

(2) $$\int_{0<|y|\leqslant 1}|y|^n|\mu|(dy)+|\mu|(1<|y|<\infty)<\infty.$$

$\rho(y)$ is an isotropic C^∞ function on R^N such that

(3) $0<\rho(y)\leqslant 1$, $|y|^{n-1}\rho(y)\leqslant 1$, $y\in R^N$, $1-\rho(y)=\mathcal{O}(|y|^{n+1})$, as $y\to 0$.

We assume that $a_\alpha\neq 0$ for some α with $|\alpha|=m$, except the case $m=0$.

The problem here is to obtain the fundamental solution $Q(t,x,\cdot)$ of

(4) $$(\partial/\partial t)u(t,x)=Au(t,x),$$

when A is essentially of elliptic type.

2. Let

$$a(z)=a_D(z)+a_I(z),$$

where

$$a_D(z)=\sum_{|\alpha|\leqslant m} i^{|\alpha|}a_\alpha z_\alpha,$$

(5) $$a_I(z)=\int_{R^N\setminus\{0\}}\left\{e^{iy\cdot z}-\rho(y)\sum_{k=0}^{n-1}1/k!\,(iy\cdot z)^k\right\}\mu(dy),\qquad y\cdot z=\sum_{j=1}^N y_jz_j.$$

The measure μ is called *degenerate*, if its support is contained in some hyperplane, which passes through the origin and has dimension at most $N-1$. μ is called *rapidly decreasing at* ∞, if, for each natural number l,

(6) $$\int_{|y|>1}|y|^l|\mu|(dy)<\infty.$$

For the second part A_I of A, we have

Theorem 1. *Let μ be given by a positive measure μ_+ as*

(7)[1] $$\mu=(-1)^{[(n-1)/2]}\mu_+.$$

1) For a real number s, $[s]$ denotes the largest integer l such that $l\leqslant s$.