128. On q-Additive Functions. II

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1. Let q be an arbitrary fixed natural number $\geqslant 2$, and $g(n)$ be a q-additive arithmetical function. In our previous paper [2], we proved a functional equation involving a q-additive function. The purpose of this article is to prove a general theorem which gives explicitly an average value of some q-additive functions as an application of our previous result.

2. In the first place, we shall mention two simple special cases of our Theorem, to clarify its nature.

(I) The relation $g(rq^k)=r$, $1\leqslant r\leqslant q-1$, $k\in N$, defines a q-additive function $g(n)$ called "sum of digits", treated in [1]. For this function, we have

$$\frac{1}{m}\sum_{n=0}^{m-1} g(n)=\frac{q-1}{2(\log q)}\log m+F\left(\frac{\log m}{\log q}\right),$$

for any $m\in N$, where $F(x)$ is a periodic function with period 1 and its Fourier coefficients are given as follows:

$$F(x)=\sum_{k\in Z} A_k\cdot\exp(2\pi ikx),$$

$$A_0=\frac{q-1}{2(\log q)}\{(\log 2\pi)-1\}-\frac{q+1}{4},$$

$$A_k=i\frac{q-1}{2\pi k}\left(\frac{2\pi ik}{\log q}+1\right)^{-1}\cdot\zeta\left(\frac{2\pi ik}{\log q}\right),\qquad k\neq 0.$$

(II) We define a 2-additive function $g(n)$ by the relation $g(2^k)=k$ for $k\in N$. Then we have

$$\frac{1}{m}\sum_{n=0}^{m-1} g(n)=\frac{1}{4(\log^2 2)}\log^2 m+(\log m)F_1\left(\frac{\log m}{\log 2}\right)+F_2\left(\frac{\log m}{\log 2}\right),$$

where $F_1(x)$ and $F_2(x)$ are periodic functions with period 1 and whose Fourier expansions are given as follows:

$$F_1(x)=\sum_{k\in Z} c_k\cdot\exp(2\pi ikx),\qquad F_2(x)=\sum_{k\in Z} d_k\cdot\exp(2\pi ikx),$$

$$c_0=\frac{(\log 2\pi)-1}{2(\log^2 2)}-\frac{1}{\log 2},$$

$$c_k=\frac{i}{2k\pi(\log 2)}\cdot\zeta\left(\frac{2\pi ik}{\log 2}\right)\cdot\left(\frac{2\pi ik}{\log 2}+1\right)^{-1},\qquad k\neq 0,$$

$$d_0=\frac{11}{24}-\frac{\zeta''(0)}{\log^2 2}-\frac{(\log 2\pi)-1}{\log 2}-\frac{(\log 2\pi)-1}{2\log^2 2},$$

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