123. *A Counterexample to a Problem on Commuting Matrices*

By Jiro SEKIGUCHI
Department of Mathematics, Tokyo Metropolitan University

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1983)

§ 1. Formulation of the problem. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and put

$$C=\{(X, Y)\in \mathfrak{g}\times\mathfrak{g}\,;\,[X, Y]=0\}.$$

It is known (cf. [4]) that C is an irreducible algebraic variety. As an easy consequence of this result, we find that

(1) $\qquad \dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)\geq \operatorname{rank}\mathfrak{g} \qquad$ for any $(X, Y)\in C$.

Here $\mathfrak{g}_X$ and $\mathfrak{g}_Y$ denote the centralizers of X and Y, respectively. Then Prof. M. Kashiwara asked the author the following

Problem. *Let $X\in\mathfrak{g}$. Then does there exist a $Y\in\mathfrak{g}$ such that Y commutes with X and that* $\dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)=\operatorname{rank}\mathfrak{g}$?

This problem connects with the study of the holonomic system of differential equations which governs an invariant eigendistribution on a real form of $\mathfrak{g}$. For the details, see [2, § 6].

The purpose of this short note is to give a counterexample to this problem when $\mathfrak{g}$ is simple of type F_4 and $X\in\mathfrak{g}$ is a certain nilpotent element.

We note here some remarks on the problem.

(1) If $X\in\mathfrak{g}$ is regular, that is, $\dim\mathfrak{g}_X=\operatorname{rank}\mathfrak{g}$, it is known (cf. [3]) that $\mathfrak{g}_X$ is abelian and therefore $\dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)=\operatorname{rank}\mathfrak{g}$ for any $Y\in\mathfrak{g}_X$.

(2) It is easy to reduce the problem to the case when X is a distinguished nilpotent element of $\mathfrak{g}$.

(3) Assume that $\mathfrak{g}$ is simple and the type of $\mathfrak{g}$ is one of $A_l, B_l, C_l, D_l, E_6, E_7, G_2$. Then for any distinguished nilpotent $X\in\mathfrak{g}$, there exists a $Y\in\mathfrak{g}$ such that Y commutes with X and $\dim\mathfrak{g}_X\cap\mathfrak{g}_Y=\operatorname{rank}\mathfrak{g}$. Namely, the problem is true in these cases. The details of this result will be published elsewhere.

(4) In the case when $\mathfrak{g}$ is simple of type E_8, the problem is rest open.

§ 2. A counterexample to the problem. Let $\mathfrak{g}$ be a simple Lie algebra of type F_4 and let X be a nilpotent element of $\mathfrak{g}$ whose weighted Dynkin diagram is $02\Rightarrow 00$ (cf. [1]).

Claim. *For any $Y\in\mathfrak{g}_X$, we have*

$$\dim(\mathfrak{g}_X\cap\mathfrak{g}_Y)\geq 6.$$