119. Boundary Value Problems for Some Degenerate Elliptic Equations of Second Order

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1. Let Ω be a bounded domain in R^N with C^∞ boundary $\partial\Omega$ and $a^{ij}(x)=a^{ji}(x)$, $b^j(x)$ and $c(x)$ be real valued functions belonging to $C^\infty(\bar{\Omega})$. In this note we shall consider the regularity up to the boundary of the solution for the following boundary value problem:

[P] $$Au=\sum_{i,j=1}^N a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{j=1}^N b^j(x)\frac{\partial u}{\partial x_j}+c(x)u=f(x) \qquad \text{in } \Omega,$$
$$u|_{\partial\Omega}=0,$$

under the assumptions on A:

A1 $a_2(x,\xi)=\sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j \geqq 0$ for $(x,\xi)\in\bar{\Omega}\times(R^N\backslash\{0\})$.

A2 $c(x)<0$ and $c^*(x)=c(x)-\sum_{j=1}^N \frac{\partial b^j}{\partial x_j}(x)+\sum_{i,j=1}^N \frac{\partial^2 a^{ij}}{\partial x_i\partial x_j}(x)<0$ on $\bar{\Omega}$.

A3 $\partial\Omega$ is non-characteristic for A.

A4 $a_1^s(x,\xi)=\sum_{j=1}^N \left\{b^j(x)-\sum_{i=1}^N \frac{\partial a^{ij}}{\partial x_i}(x)\right\}\xi_j\neq 0$

for $(x,\xi)\in\sum=\{(x,\xi)\in\bar{\Omega}\times(R^N\backslash\{0\})\,|\,a_2(x,\xi)=0\}$.

Several existence, uniqueness and regularity theorems of the problem [P] were proved in Fichera [1], [2], Kohn-Nirenberg [4] and Oleinik [5], Oleinik-Radkevič [6]. In fact, it is known that there is a uniquely determined weak solution $u\in L^2(\Omega)$ of [P] with $f\in L^2(\Omega)$ if the conditions A1, A2 and A3 hold. Here $u\in L^2(\Omega)$ is called the weak solution of [P] with $f\in L^2(\Omega)$ if the identity

(1.1) $\int_\Omega u\overline{A^t v}dx=\int_\Omega f\bar{v}dx$ holds for all $v\in C^\infty(\bar{\Omega})$ with $v|_{\partial\Omega}=0$,

where
$$A^t v=\sum_{i,j=1}^N a^{ij}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}-\sum_{j=1}^N\left\{b^j(x)-2\sum_{i=1}^N\frac{\partial a^{ij}}{\partial x_i}(x)\right\}\frac{\partial v}{\partial x_j}+c^*(x)v.$$

Concerning the local regularity of this weak solution, we can apply Theorem 5.9 in Hörmander [3] to the operator A if the conditions A1 and A4 hold (see also Radkevič [7]). That is, if u is the weak solution of [P] with $f\in H^k(\Omega)$,[1] then we have $u\in H^{k+1}(U)$ for any open set U such that $\bar{U}\subset\Omega$. This is the reason why we consider the boundary value problem [P].

1) $H^k(\Omega)$ denotes the Sobolev space on Ω for non-negative integer k.