13. Non-Uniqueness in the Cauchy Problem for Partial Differential Operators with Multiple Characteristics

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The purpose of this paper is to give a necessary condition for uniqueness of C^∞-solutions of the non-characteristic Cauchy problem for a class of partial differential operators with multiple characteristics of constant multiplicity or of variable multiplicity. For operators with multiple characteristics, many authors obtained various sufficient conditions for uniqueness on the lower order terms (see [1], [4], [5], [7]). However, except [1], it is still unclear whether their conditions are necessary or not. We shall give an answer to this question. The author believes that our results will clarify a role of lower order terms in this theory.

§1. **Statement of results.** We consider the following operator in $\boldsymbol{R}^{d+1}$:

$$P=P(t, x\,;\partial_t, D_x)=\partial_t^p+t^k A(t, x\,; D_x)-t^m B(t, x\,; D_x),$$

where $\partial_t=\partial/\partial t$, $D_x=(\partial/i\partial x_1, \cdots, \partial/i\partial x_d)$, $k, m\in \boldsymbol{N}$, A and B are partial differential operators with respect to x of order q and $q-r$ respectively ($p\geqq q>r\geqq 1$) with C^∞-coefficients in U, an open neighborhood of the origin in $\boldsymbol{R}^{d+1}$. Let $A_q(t, x\,;\xi)$ and $B_{q-r}(t, x\,;\xi)$ be the principal symbols of A and B respectively. We assume

(A.1) $$k>(pr+qm)/(q-r).$$

We also assume that there exist $\xi^0\in \boldsymbol{R}^d\backslash\{0\}$ and a root $C=C(\xi^0)$ of the equation $X^p=B_{q-r}(0,0\,;\xi^0)-A_q(0,0\,;\xi^0)$ satisfying

(A.2) $$\operatorname{Re} C(\xi^0)>0,$$

(A.3) $$\operatorname{Re}\left\{\left(\frac{A_q(0,0\,;\xi^0)}{B_{q-r}(0,0\,;\xi^0)-A_q(0,0\,;\xi^0)}+1-\frac{q}{r}\right)C(\xi^0)\right\}>0.$$

Now, we state the main theorem.

Theorem 1. *Under assumptions* (A.1)–(A.3), *there exist an open neighborhood U' of the origin and $u, f\in C^\infty(U')$ satisfying*

$$Pu-fu=0, \qquad (0,0)\in \operatorname{supp} u\subset\{t\geqq 0\}.$$

Next, we consider the following operator:

$$P'=P+\sum_{i+j\leqq p}{}' t^{k(i,j)}B_{i,j}(t, x\,; D_x)\partial_t^j,$$

where P is the operator treated above and $B_{i,j}$ are operators of order i with coefficients in $C^\infty(U)$. We assume