116. On Compact Kähler Manifolds of Constant Scalar Curvatures

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1. Introduction. The purpose of this note is to generalize the result obtained in [1]. To be more precise we shall present an obstruction to the existence of a Kähler metric of constant scalar curvature in any fixed Kähler class of a compact complex manifold M with $b_1(M)=0$.

Let $\omega=(i/2\pi)\ g_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$ be a Kähler form of M, $\gamma_\omega=-(i/2\pi)\ \partial\bar\partial \log\det(g_{\alpha\bar\beta})$ the Ricci form of ω, and τ_ω the harmonic part of $\gamma_\omega-\omega$. Then there exists a real valued smooth function F_ω, uniquely determined up to any constant function, such that

$$\gamma_\omega=\omega+\tau_\omega+\frac{i}{2\pi}\partial\bar\partial F_\omega.$$

We denote by $\mathfrak{h}(M)$ the complex Lie algebra of all holomorphic vector fields of M. We define a linear function $f_{[\omega]}$ of $\mathfrak{h}(M)$ into $\boldsymbol{C}$ by

$$f_{[\omega]}(X)=\int_M XF_\omega\omega^m$$

where $m=\dim M$.

Theorem 1. *Let M be a compact complex manifold with $b_1(M)=0$ admitting Kählerian structures. Then the function $f_{[\omega]}$ depends only on the Kähler class $[\omega]\in H^2(M;\boldsymbol{R})$. If M admits a Kähler form $\tilde\omega\in[\omega]$ of constant scalar curvature, then $f_{[\omega]}=0$.*

Theorem 2. *The function $f_{[\omega]}$ is invariant under the group G of all holomorphic transformations of M preserving the class $[\omega]$. In particular the derived algebra of $\mathfrak{h}(M)$ is contained in the kernel of $f_{[\omega]}$ and $f_{[\omega]}$ is a Lie algebra homomorphism. If $\mathfrak{h}(M)$ is semisimple then $f_{[\omega]}=0$. If $f_{[\omega]}\neq 0$ then $\mathfrak{h}(M)$ contains a hyperplane invariant under G.*

If the first Chern class $c_1(M)$ is positive, any Kähler form of constant scalar curvature in $c_1(M)$ is Einstein. So the result of this paper generalizes that of [1].

We remark that $f_{[\omega]}$ actually varies as ω does; this can be observed by considering a product of two compact Kähler manifolds M_i with the Kähler form ω_i, $i=1,2$, such that $f_{[\omega_i]}\neq 0$ and taking Kähler forms $\omega=k_1\omega_1+k_2\omega_2$ on $M_1\times M_2$ for positive parameters k_1 and k_2.

2. Proof. Fix a Kähler form ω_0. Any Kähler form ω_1 cohomologous to ω_0 can be joined by a smooth family of Kähler forms $\omega_t=\omega_0$