114. A Note on the Approximate Functional Equation for $\zeta^2(s)$

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1. Let $0\leq\sigma\leq 1$, $t\geq 2$, $XY=t^2/4\pi^2$, $xy=t/2\pi$, and put

$$\zeta(s)=\sum_{n\leq x} n^{-s}+\chi(s)\sum_{n\leq y} n^{s-1}+E(s,x), \tag{1}$$

$$\zeta^2(s)=\sum_{n\leq X} d(n)n^{-s}+\chi^2(s)\sum_{n\leq Y} d(n)n^{s-1}+D(s,X)$$

where d is the divisor function and $\chi(s)=2^s\pi^{s-1}\sin(s\pi/2)\Gamma(1-s)$. Hardy and Littlewood [1] proved that

$$E(s,x)\ll x^{-\sigma}+t^{1/2-\sigma}y^{\sigma-1} \tag{2}$$

as well as

$$D(s,X)\ll X^{1/2-\sigma}\left(\frac{X+Y}{t}\right)^{1/4}\log t.$$

Later Titchmarsh [5] replaced the latter by

$$D(s,X)\ll (X+Y)^{1/2-\sigma}\log t. \tag{3}$$

Also we should note that Jutila [4] remarked recently that (3) is a consequence of Voronoi's summation formula.

The arguments of these authors are rather elaborated, mainly because they treated $\zeta^2(s)$ directly, i.e. without recoursing to the known approximations for $\zeta(s)$. Here we shall show that if we make use of Dirichlet's device:

$$\sum_{n\leq N} d(n)a_n=2\sum_{n\leq\sqrt{N}}\sum_{m\leq N/n} a_{nm}-\sum_{n\leq\sqrt{N}}\sum_{m\leq\sqrt{N}} a_{nm}, \tag{4}$$

then, as far as the most interesting case $X=Y=t/2\pi$ is concerned, (3) is a quite simple consequence of (2).

For this end let $U=t/2\pi$, $u=\sqrt{U}$. Then by (4) we have

$$\begin{aligned}\sum_{n\leq U} d(n)n^{-s}&+\chi^2(s)\sum_{n\leq U} d(n)n^{s-1}\\ &=2\sum_{m\leq u} m^{-s}\sum_{n\leq U/m} n^{-s}+2\chi^2(s)\sum_{m\leq u} m^{s-1}\sum_{n\leq U/m} n^{s-1}\\ &\quad-\Big(\sum_{m\leq u} m^{-s}\Big)^2-\chi^2(s)\Big(\sum_{m\leq u} m^{s-1}\Big)^2.\end{aligned}$$

And by (1) this is equal to

$$\begin{aligned}2\sum_{m\leq u} m^{-s}&\Big\{\zeta(s)-\chi(s)\sum_{n\leq m} n^{s-1}-E(s,U/m)\Big\}\\ &+2\chi^2(s)\sum_{m\leq u} m^{s-1}\Big\{\zeta(1-s)-\chi(1-s)\sum_{n\leq m} n^{-s}-E(1-s,U/m)\Big\}\\ &+2\chi(s)\sum_{m\leq u} m^{-s}\sum_{n\leq u} n^{s-1}-(\zeta(s)-E(s,u))^2.\end{aligned}$$

Then, after some rearrangement, we get