# 113. Construction of Certain Real Quadratic Fields

By Tsuyoshi UEHARA
Department of Mathematics, Saga University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1983)

Let $n$ be a given natural number. In this note we shall construct real quadratic fields whose fundamental units are congruent to $\pm 1$ modulo $n$. We also give a new proof of the existence of infinitely many real quadratic fields each with class number divisible by $n$ (cf. Weinberger [3], Yamamoto [4]).

Let $Z$, $Q$ be the ring of rational integers, the field of rational numbers respectively. For a rational integer $m \neq 0$ and a prime $p$ we denote by $\mathrm{ord}_p\, m$ the greatest nonnegative rational integer $f$ such that $m \equiv 0 \pmod{p^f}$.

**Lemma.** *Let $\alpha$, $\beta$ be integers of a quadratic field $K$ such that $\alpha = \pm\beta^n$ for some $n>1$ in $Z$. We write $\alpha=(a+b\sqrt{d})/2$, $\beta=(s+t\sqrt{d})/2$ with $a$, $b$, $s$, $t$ in $Z$, where $d$ is the discriminant of $K$. If $p$ is a prime dividing $d$ such that $\mathrm{ord}_p\, a=\mathrm{ord}_p\, 2$, then we have*

$$\mathrm{ord}_p\, t = \mathrm{ord}_p\, b - \mathrm{ord}_p\, n$$

*except in the following two cases*: (i) $p=2$, $\mathrm{ord}_2\, d=2$ *and* $n \equiv 0 \pmod 2$, (ii) $p=3$, $d \equiv 6 \pmod 9$ *and* $n \equiv 0 \pmod 3$.

*Proof.* First assume that $\mathrm{ord}_p\, d = \mathrm{ord}_p\,(4p)$. Then $\mathrm{ord}_p\, a=\mathrm{ord}_p\, 2$ implies $\mathrm{ord}_p\, s = \mathrm{ord}_p\, 2$. If $5 \leqq k \leqq n$, we have $\mathrm{ord}_p \binom{n}{k} \geqq \mathrm{ord}_p\, n - \mathrm{ord}_p\, k \geqq \mathrm{ord}_p\, n + 1 - k/2$. Hence

$$b \equiv \pm nt(s/2)^{n-3}\{(s/2)^2+(n-1)(n-2)t^2d/24\} \pmod{p^{g+1}}$$

with $g=\mathrm{ord}_p\,(nt)$. Thus $\mathrm{ord}_p\, b = g$ holds except in the case (ii). Next let $p=2$, $\mathrm{ord}_2\, d=2$ and $(n,2)=1$. Then $\beta^2 \equiv 0$ or $1 \pmod 2$ according as $s/2 \equiv t$ or $t+1 \pmod 2$. Since $\mathrm{ord}_2\, a=1$, $\alpha \equiv \beta \pmod 2$ and $s/2 \equiv t+1 \equiv 1 \pmod 2$. Hence $b \equiv \pm nt(s/2)^{n-1} \pmod{2t}$. Thus the lemma follows.

**Theorem.** *Let $n$ be a given natural number and let $k>1$ be a square free rational integer such that $k \equiv 0 \pmod p$ for any prime $p$ dividing $n$. We put*

$$\varepsilon = (kn^2 \pm 2 + n\sqrt{m})/2 \quad \textit{with} \quad m = k(kn^2 \pm 4),$$

*and assume that $kn^2 \pm 4 \neq c^2$, $2c^2$ for any $c$ in $Z$ and that $m \equiv 3 \pmod 9$ if 3 divides $n$. Then $\varepsilon>1$ is the fundamental unit of $K=Q(\sqrt{m})$.*

*Proof.* It is easy to see that $\varepsilon>1$ is a unit of $K$ with norm 1. We write $kn^2 \pm 4 = c^2u$ with $c$ in $Z$ and a square free rational integer $u>0$. From the assumption we have $u \geqq 3$. Since $(u,k)=1$ or 2, the discriminant $d$ of $K$ is $ku$ if $n$ is odd, and is $4ku$ if $n$ is even. Note that