107. A Truncated Cube Functional Equation

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§1. **Introduction.** The purpose of this note is to announce some equivalence relations among certain particular polyhedral mean value type functional equations without any regularity assumptions.

Let $(G, +)$ be an Abelian group in which it is possible to divide by 2, and let F be a field of characteristic zero. For a function $f: G\times G \times G\to F$ we define the shift operators X_1^t, X_2^t, and X_3^t by $(X_1^t f)(x,y,z) = f(x+t,y,z)$, $(X_2^t f)(x,y,z)=f(x,y+t,z)$, and $(X_3^t f)(x,y,z)=f(x,y,z+t)$ for all $x, y, z, t\in G$. In particular $1=X_1^0=X_2^0=X_3^0$ denotes the identity operator. We note that the ring of linear transformation generated by this family of transformations is commutative and distributive.

L. Etigson [2] and L. Sweet [5] considered the equivalence of the following cube and octahedron mean value functional equations, which are the most fundamental particular polyhedral mean value type functional equations, under the assumption $f: G\times G\times G\to F$:

(1.1) $$(C(t)f)(x,y,z)=8f(x,y,z),$$

(1.2) $$(O(t)f)(x,y,z)=6f(x,y,z)$$

where the operators $C(t)$ and $O(t)$ are defined by

$$C(t)=\prod_{i=1}^{3}(X_i^t+X_i^{-t}) \quad \text{and} \quad O(t)=\sum_{i=1}^{3}(X_i^t+X_i^{-t}).$$

In this note we will consider the equivalence of (1.1) and the polyhedral mean value functional equation

(1.3) $$(T(t)f)(x,y,z)=12f(x,y,z)$$

where the operator $T(t)$ is defined by

$$T(t)=(X_1^t+X_1^{-t})(X_2^t+X_2^{-t})+(X_2^t+X_2^{-t})(X_3^t+X_3^{-t})+(X_3^t+X_3^{-t})(X_1^t+X_1^{-t}).$$

By a geometric interpretation we call equation (1.3) a *truncated cube mean value functional equation.*

§2. **Equivalence of (1.1) and (1.3). Theorem 1.** *If a function $f: G\times G\times G\to F$ satisfies equation* (1.1) *for all $x, y, z, t\in G$, then also* (1.3) *for all $x, y, z, t\in G$ and conversely so that* (1.1) *and* (1.3) *are equivalent.*

By using the operator notations in §1 we have $C(2t)=\prod(X_i^{2t}+X_i^{-2t})$ and readily obtain

(i) $C(t)^2=(C(t))(C(t))=C(2t)+2T(2t)+4O(2t)+8,$

(ii) $O(t)^2=(O(t))(O(t))=O(2t)+2T(t)+6,$