103. On Poles of the Rational Solution of the Toda Equation of Painlevé-II Type

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§1. **Summary.** The Toda equation

(1.1) $$q_n' = p_{n-1} - p_n, \quad p_n' = p_n(q_n - q_{n+1}), \quad n = 0, \pm 1, \pm 2, \cdots$$

admits the special rational solution

(1.2) $$q_n = (\log P_n/P_{n+1})', \qquad p_n = (\log P_{n+1})'' - t/4$$

where

(1.3) $$P_n = \prod_{k=1}^{d(n)} (t - a_{n,k}) = \sum_{j=0}^{f(n)} P_{n,j} t^{d(n)-3j}$$

are the polynomials of degree $d(n) = n(n-1)/2$ with integral coefficients ($P_{n,0} = 1$, $P_{n,f(n)} \neq 0$, $f(n) = [n(n-1)/6]$). These polynomials were introduced by A. I. Yablonskii [1] and A. P. Vorobiev [2] who showed that q_n satisfies the Painlevé-II equation

(1.4) $$q_n'' = 2q_n^3 + tq_n + n.$$

All zeros of P_n are simple, P_n and P_{n+1} have no common zero. So q_n has n^2 simple poles and p_n has $n(n+1)/2$ double poles.

A sharp estimate for the maximal modulus of these poles is obtained. $A_n = \max\{|a_{n,k}|; 1 \le k \le d(n)\}$ satisfies

(1.5) $$n^{2/3} \le A_{n+2} \le 4n^{2/3} \qquad n = 0, 1, 2, \cdots.$$

§2. **Recurrence relation.** If we define the rational functions q_n and p_n by the recurrence relation

(2.1) $$q_0 = 0, \qquad p_0 = -t/4,$$

(2.2) $$q_n = (2n-1)/4p_{n-1} - q_{n-1}, \qquad p_n = -(p_{n-1} + q_n^2 + t/2),$$

(2.3) $$q_{-n} = -q_n, \quad p_{-n} = p_{n-1}, \quad n = 1, 2, 3, \cdots$$

then

Theorem 2.1. *$\{q_n, p_n\}$ satisfies the Toda equation* (1.1), *q_n satisfies the Painlevé*-II *equation* (1.4) *and p_n satisfies*

(2.4) $$p_n p_n'' - p_n'^2/2 + 4p_n^3 + tp_n^2 + (2n+1)^2/32 = 0$$

for every integral n.

§3. **Yablonskii-Vorobiev's polynomials.** The rational functions P_n are determined uniquely by the relation

(3.1) $$p_n = -P_n P_{n+2}/4P_{n+1}^2, \qquad n = 0, \pm 1, \pm 2, \cdots$$

with initial condition

(3.2) $$P_0 = P_1 = 1.$$

Integrating the Toda equation with respect to n we have (1.2). So