99. On Hilbert Modular Forms. III

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The purpose of this note is to give a final result on a problem posed in [3], which is concerned with the structure of the ring of Hilbert modular forms with integral Fourier coefficients. Let K be a real quadratic field and denote by $A_Z(\Gamma_K)_k$ the Z-module of symmetric Hilbert modular forms of weight k with integral Fourier coefficients. We put

$$A_Z(\Gamma_K)=\bigoplus_{k\geqq 0} A_Z(\Gamma_K)_{2k}, \qquad A_Z^a(\Gamma_K)=\bigoplus_{k\geqq 0} A_Z(\Gamma_K)_k.$$

Then $A_Z(\Gamma_K)$ is a graded subring of $A_Z^a(\Gamma_K)$. In [3], the author showed that the ring $A_Z(\Gamma_{Q(\sqrt{2})})$ is generated by three forms V_2, V_4 and V_6 over Z and $A_Z(\Gamma_{Q(\sqrt{5})})$ is generated by four forms W_2, W_6, W_{10} and W_{12}, where the subscripts denote the weight, and these modular forms are explicitly expressed by Eisenstein series (cf. [3], [4]).

In [5], H. L. Resnikoff showed the existence of a symmetric Hilbert modular form of odd weight 15 for $Q(\sqrt{5})$ by using Igusa-Hammond's modular imbedding, and he gave a quadratic relation it satisfies. We can show that Resnikoff's method is applicable in the case $K=Q(\sqrt{2})$.

From now on, we restrict ourselves to the case $K=Q(\sqrt{2})$. In this case, every element $f(\tau)$ in $A_Z^a(\Gamma_K)$ has the following Fourier expansion.

$$\begin{aligned} f(\tau)= & \sum_{\substack{\nu\gg 0 \\ \nu\equiv 0 \ (\mathrm{mod}\ 1/2\sqrt{2})}} a_f(\nu)\exp[2\pi i tr(\nu\tau)] \\ = & a_f(0)+a_f((-1+\sqrt{2})/2\sqrt{2})x^{-1}q+a_f(1/2)q \\ & +a_f((1+\sqrt{2})/2\sqrt{2})xq+a_f((-2+2\sqrt{2})/2\sqrt{2})x^{-2}q^2 \\ & +a_f((-1+2\sqrt{2})/2\sqrt{2})x^{-1}q^2+a_f(1)q^2 \\ & +a_f((1+2\sqrt{2})/2\sqrt{2})xq^2+a_f((2+2\sqrt{2})/2\sqrt{2})x^2q^2 \\ & +\cdots, \end{aligned}$$

where $\tau=(z_1,z_2)\in\mathfrak{H}\times\mathfrak{H}$, $q=\exp[\pi i(z_1+z_2)]$, $x=\exp[\pi i(z_1-z_2)]$. We denote by $G_k(\tau)$ the normalized Eisenstein series for the Hilbert modular group $\Gamma_K=SL(2,\mathfrak{o}_K)$. We put

$$\begin{aligned} & H_2=G_2, \qquad H_4=2^{-6}\cdot 3^{-2}\cdot 11(G_2^2-G_4), \\ & H_6=-2^{-8}\cdot 3^{-3}\cdot 13^{-1}\cdot 5\cdot 7^2G_2^3+2^{-8}\cdot 3^{-2}\cdot 5^{-1}\cdot 13^{-1}\cdot 11\cdot 59G_2G_4 \\ & \qquad -2^{-7}\cdot 3^{-3}\cdot 5^{-1}\cdot 13^{-1}\cdot 19^2G_6. \end{aligned}$$

If we use the notation in [3], then

$$H_2=V_2, \quad H_4=V_4, \quad H_6=V_6-V_2V_4.$$