95. On Approximation by Integral Müntz Polynomials

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In 1914 there appeared two independent articles of importance on Weierstrass' approximation theorem. Kakeya [6] considered approximation of a given continuous function $f(x)$ on $[a, b]$ by polynomials with integral coefficients, while Müntz [7] studied the condition on the sequence $\Lambda=\{\lambda_n\}$ $(0=\lambda_0<\lambda_1<\cdots<\lambda_n)$ to approximate $f(x)$ by the "Müntz polynomials"

$$p(x)=\sum_{k=0}^{n} a_k x^{\lambda_k}, \tag{1}$$

where the coefficients a_k's are real.

Kakeya proved that on $[0, 1]$ $f(x)$ is uniformly approximated by integral polynomials iff $f(0)$ and $f(1)$ are both integers, and showed that if $\alpha \geqq 4$, $f(x)$ cannot be uniformly approximated on $[0, \alpha]$ by integral polynomials unless it is such a polynomial.

The necessary and sufficient condition found by Müntz was

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}=+\infty, \tag{2}$$

which is now usually called "Müntz condition". Their aspects and results have been both unified and extended recently (cf. Ferguson [2] for basic results). One of the fundamental problems is to find conditions to approximate $f(x)$ on $[0, \alpha]$ by integral Müntz polynomials, i.e. $p(x)$ with integer coefficients a_k's.

If we denote by $C_0[0, \alpha]$ the set of all continuous functions $f(x)$ on $[0, \alpha]$ such that $f(m)$ is integer for any integer m in $[0, \alpha]$, then Ferguson and Golitschek [3] proved that when Λ is a sequence of positive integers and $\alpha \leqq 1$, (2) is the necessary and sufficient condition for $f \in C_0[0, \alpha]$ being uniformly approximated by integral Müntz polynomials ([2], Chap. 8). Later Golitschek [4] has succeeded in proving this true for any $\lambda_n \uparrow \infty$. Also Ferguson [1] showed, among other things, that the assertion becomes false if $\alpha > 1$.

Now define for the increasing sequence Λ of positive numbers,

$$\underline{D}(\Lambda)=\liminf_{N\to\infty} \frac{N}{\lambda_N}, \qquad \bar{D}(\Lambda)=\limsup_{N\to\infty} \frac{N}{\lambda_N},$$

which are called respectively the lower and the upper asymptotic densities of Λ. If $\underline{D}(\Lambda)=\bar{D}(\Lambda)<\infty$, we denote it by