11. *Integrality of Certain Algebraic Values Attached to Modular Forms*

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(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 12, 1983)

Introduction. In this note we report integrality results on some algebraic numbers appearing in the theory of modular forms: Quotients of Petersson inner products for elliptic modular forms, special values of the "second" L-function attached to elliptic eigen cusp forms, and Fourier coefficients of generalized Eisenstein series of degree two in the sense of Langlands and Klingen. Details will appear elsewhere. For motivations, we refer to Kurokawa [4], [5] and [6], [8], [9]. The author would like to thank Prof. N. Kurokawa for encouragements.

§ 1. **Petersson inner products.** For integers $n\geqq 1$ and $k\geqq 0$, we denote by $M_k(\Gamma_n)$ (or $S_k(\Gamma_n)$) the vector space over the complex number field $\boldsymbol{C}$ consisting of all Siegel modular (or cusp) forms of degree n and weight k. Each $F\in M_k(\Gamma_n)$ has a Fourier expansion of the form: $F=\sum_{T\geqq 0} a(T,F)q^T$, where $q^T=\exp(2\pi\sqrt{-1}\,\mathrm{trace}\,(TZ))$ with a variable Z on the Siegel upper half space of degree n, and T runs over all $n\times n$ symmetric positive semi-definite semi-integral matrices. For a subring R of $\boldsymbol{C}$, we put $M_k(\Gamma_n)_R=\{F\in M_k(\Gamma_n)\,|\,a(T,F)\in R \text{ for all } T\geqq 0\}$ and $S_k(\Gamma_n)_R=S_k(\Gamma_n)\cap M_k(\Gamma_n)_R$ (R-modules).

In the first two sections, we treat elliptic modular forms. For each integer $m\geqq 1$, let $T(m): M_k(\Gamma_1)\to M_k(\Gamma_1)$ be the m-th Hecke operator and $\boldsymbol{T}_{\boldsymbol{Q}}=\boldsymbol{Q}[T(m)\,|\,m\geqq 1]$ be the Hecke algebra over the rational number field $\boldsymbol{Q}$. For $F\in S_k(\Gamma_1)$ and $G\in M_k(\Gamma_1)$ we put

$$\langle F, G\rangle=\frac{3}{\pi}\int_{\mathfrak{F}} F(z)\overline{G(z)}y^{k-2}dxdy \qquad (z=x+\sqrt{-1}\,y)$$

where $\mathfrak{F}$ is a fundamental domain of $\Gamma_1\backslash\mathfrak{H}_1$, $\mathfrak{H}_1$ being the upper half plane.

Let $\bar{\boldsymbol{Q}}$ be the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$, and let $\bar{\boldsymbol{Z}}$ be the ring of algebraic integers in $\bar{\boldsymbol{Q}}$. For each non-zero $\alpha\in\bar{\boldsymbol{Q}}$, we denote by $\mathrm{D}(\alpha)$ the minimal positive $M\in\boldsymbol{Z}$ $(=\bar{\boldsymbol{Z}}\cap\boldsymbol{Q})$ such that $M\alpha\in\bar{\boldsymbol{Z}}$, and we put $\mathrm{Num}\,(\alpha)=\mathrm{D}(\alpha)\alpha$.

Now let $f=\sum_{n\geqq 1} a(n)q^n\in S_k(\Gamma_1)$ be a normalized eigen cusp form, i.e., $a(1)=1$ and $T(n)f=a(n)f$ for all $n\geqq 1$. Let $\boldsymbol{Q}(f)=\boldsymbol{Q}(a(n)\,|\,n\geqq 1)$ be the totally real number field generated by the eigen values of f, and let $\mathfrak{D}(\boldsymbol{Q}(f))$ be the different of $\boldsymbol{Q}(f)/\boldsymbol{Q}$. We put $\boldsymbol{Z}(f)=\bar{\boldsymbol{Z}}\cap\boldsymbol{Q}(f)$, and denote by κ the exponent of the finite abelian group $\boldsymbol{Z}(f)/\boldsymbol{Z}[a(n)\,|\,n\geqq 1]$.