81. *The Thue-Siegel-Roth Theorem for Values of Algebraic Functions*

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(Communicated by Heisuke HIRONAKA, M. J. A., June 14, 1983)

§0. The Thue-Siegel-Roth theorem [1], [2] on the approximation of algebraic numbers by rational numbers states that for a given algebraic number α and any $\varepsilon>0$ there exist only finitely many rational approximations p/q to α such that $|\alpha-p/q|<|q|^{-2-\varepsilon}$. Here, as well as in earlier versions [3], the results are ineffective; they depend on the knowledge of at least one good approximation to α. Such good approximations are known only for an α of a special form, see [4] and [5]. Here we prove an entirely new theorem on approximations of algebraic numbers by considering diophantine approximations to values of algebraic functions. Our result shows that for an algebraic functions defined over $\boldsymbol{Q}(x)$ and regular at $x=0$, the "Roth" theorem holds for the number $f(r)$ with a rational $r\neq 0$ close to 0. Our methods are based on the Wronskian technique developed in [6] for the functional version of Roth's theorem. A complete proof is presented for cubic algebraic functions, satisfying Ricatti equations.

§1. Let $f(x)$ be an algebraic function over $\boldsymbol{Q}(x)$ defined as a solution of an algebraic equation $P(x, f(x))=0$ for an absolutely irreducible polynomial $P(x, y)$ over $\boldsymbol{Q}[x, y]$. We also assume that $f(x)$ is regular at $x=0$ and has the Taylor expansion $f(x)=\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \boldsymbol{Q}$.

Theorem 1. *Let $f(x)$ be as above and let $r=a/b$ for rational integers a and b. For every $\varepsilon>0$ there exist effective constants $c_1=c_1(\varepsilon, f)>0$ and $c_2=c_2(\varepsilon, a, b, f)$ with the following properties. If $|b|^{\varepsilon}\geq c_1\cdot|a|^{2(1+\varepsilon)}$, then*

$$\left|f(r)-\frac{P}{Q}\right|>|Q|^{-2-\varepsilon}$$

for arbitrary relatively prime rational integers P, Q with $|Q|\geq c_2$.

A similar result holds in the p-adic metric, if one replaces $|b|$ with $|b|_p$ and $|f(r)-P/Q|$ by $|f(r)-P/Q|_p$. The proof of this theorem is based on the author's studies of generalizations of Padé approximations to solutions of linear differential equations and their relation to the Ricatti equation and Wronskian calculus [6]. Theorem 1 above holds for an arbitrary (G, C)-function $f(x)$ (see the definition in [7]). Moreover, these results can be generalized for the simultaneous