76. *Class Numbers of Pure Cubic Fields*

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Let n be a given natural number. It was proved by Uchida [2] that there exist infinitely many cubic cyclic fields whose class numbers are divisible by n. On the other hand, by the main result in Azuhata-Ichimura [1], one can see that it is true for cubic fields with only one real prime spot. In this note we shall show that it is also true for pure cubic fields, that is $\boldsymbol{Q}(\sqrt[3]{a})$ for some $a \in \boldsymbol{Z}$. Our goal is the following

Theorem. *For a given natural number n, there exist infinitely many pure cubic fields whose class numbers are divisible by n.*

To prove this, we will use the following notations. For an arbitrary number field k of finite degree, $k^{\times}$ denotes its multiplicative group. For a natural number ν and a prime ideal $\mathfrak{p}$ of k satisfying $N\mathfrak{p} \equiv 1 \pmod{\nu}$ (where $N\mathfrak{p}$ is the absolute norm of $\mathfrak{p}$), $(\ /\mathfrak{p})_{\nu}$ denotes the ν-th power residue mod $\mathfrak{p}$. Let n be a fixed natural number greater than 1, S be the set of all prime factors of n. Set $m=\prod_{l \in S} l$ and $m_0=m$, $2m$, $3m$ or $6m$ according to $m\equiv 3, 0, \pm 1$ or $\pm 2 \pmod 6$. Let F be the cyclotomic field of the m_0-th roots of unity, ζ be a fixed primitive cubic root of unity i.e. $\zeta^3=1$, $\zeta \neq 1$.

Lemma. *There exist infinitely many prime ideals $\mathfrak{p}$ of F of degree* 1 *satisfying*

$$\left(\frac{\zeta-1}{\mathfrak{p}}\right)_l \neq 1, \quad \left(\frac{\zeta}{\mathfrak{p}}\right)_l = 1 \qquad \text{for all } l \in S.$$

Proof. Let $\mathfrak{l}$ be a prime ideal of F lying above 3. As $3 \| m_0$, $\mathfrak{l}$ is unramified for $F/\boldsymbol{Q}(\zeta)$ and consequently the order of $\zeta-1$ at $\mathfrak{l}$ is 1. Therefore $\zeta-1$ cannot be equal to $\alpha^l\zeta^c$ for any $\alpha \in F^{\times}$, $l \in S$ and $c \in \boldsymbol{Z}$. This implies that the extension $F(\sqrt[m]{\zeta-1})/F$ is cyclic of degree m and that $F(\sqrt[m]{\zeta-1}) \cap F(\sqrt[m]{\zeta})=F$. So, by the density theorem, we can choose infinitely many prime ideals $\mathfrak{p}$ of F of degree 1 which are inert for $F(\sqrt[m]{\zeta-1})$, while decomposed completely for $F(\sqrt[m]{\zeta})$. Our lemma follows immediately from this.

We now prove the theorem. By the above lemma, we can find two prime ideals $\mathfrak{p}$, $\mathfrak{q}$ of F of degree 1 satisfying

$$(1) \quad \left(\frac{\zeta-1}{\mathfrak{p}}\right)_l \neq 1, \ \left(\frac{\zeta-1}{\mathfrak{q}}\right)_l \neq 1, \ \left(\frac{\zeta}{\mathfrak{p}}\right)_l = 1, \ \left(\frac{\zeta}{\mathfrak{q}}\right)_l = 1 \qquad \text{for } l \in S.$$