74. *Asymptotic Error Estimation for Spline-on-Spline Interpolation*

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1. Introduction and description of method. We shall consider an asymptotic error estimation for spline-on-spline interpolation on a uniform mesh. The spline-on-spline technique is described in [3]: Suppose we have an interval [0, 1] partitioned in n equal parts of length h. In order to find for a given function $f(x)$ an approximant of its first derivative $f'(x)$, we interpolate $f(x)$ by a cubic spline $s(x)$ over the mesh under appropriate additional end conditions at the endpoints. Then replacing $f(x)$ by $s'(x)$, we get an approximant of $f''(x)$, and this procedure can be continued. Ahlberg *et al.* [1] observed that it gives excellent results for the second derivative of sin x, and Dolezal and Tewarson [3] obtained error bounds for the spline-on-spline interpolation. In what follows, let r, k and l be nonnegative integers, and $s_i^{(j)}=s^{(j)}(ih)$.

Corresponding to the sufficiently smooth function $f(x)$ on [0, 1], one constructs a cubic spline $s(x)$ of the form

$$(1)\qquad s(x)=\sum_{i=-3}^{n-1}\alpha_i Q_4(x/h-i),\qquad nh=1$$

so that

$$(2)\qquad s_i=f_i,\qquad i=0,1,\cdots,n.$$

Since s depends upon $n+3$ parameters, two additional conditions are required toward the determination of s. Under various (end) conditions in [5], we have

$$(3)\qquad f_i'-s_i'=(h^4/180)f_i^{(5)}+O(h^5),\qquad i=0,1,\cdots,n.$$

In the present paper we take these to be homogeneous end conditions:

$$(4)\qquad \Delta^r s_0'=\nabla^r s_n'=0$$

where Δ and ∇ are forward and backward difference operators, respectively. By the consistency relation [1, p. 13] and (2), we have

$$(5)\qquad (1/6)(s_{i+1}'+4s_i'+s_{i-1}')=(1/2)h^{-1}(s_{i+1}-s_{i-1})$$
$$=(1/2)h^{-1}(f_{i+1}-f_{i-1}),\quad i=1,2,\cdots,n-1.$$

By (5), end condition $\Delta^r s_0'=0$ may be equivalently rewritten as follows

$$(6)\qquad s_0'+a_r s_1'=L_r(f_0,f_1,\cdots,f_r),\qquad r\neq 2$$

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