8. Some New Linear Relations for Odd Degree Polynomial Splines at Mid-Points

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By making use of the B-spline $Q_{p+1}(x)$:

$$Q_{p+1}(x)=(1/p!)\sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}(x-i)_+^p,$$

we consider the spline function $s(x)$ of the form

$$s(x)=\sum_{i=-p}^{n}\alpha_i Q_{p+1}\left(\frac{x}{h}-i+\frac{1}{2}\right), \qquad nh=1$$

where

$$(x-i)_+^p=\begin{cases}(x-i)^p & \text{for } x\geq i\\ 0 & \text{for } x<i.\end{cases}$$

It is well known that

(i) s is a polynomial of degree p on $\left[\left(i-\frac{1}{2}\right)h,\ \left(i+\frac{1}{2}\right)h\right]$,

(ii) $s\in C^{p-1}(-\infty,\infty)$.

Let p and k be integers such that $1\leq k\leq p-1$, then the following consistency relation holds

$$h^{-k}\left\{Q_{p+1}^{(k)}\left(p+1-\frac{1}{2}\right)s_i+Q_{p+1}^{(k)}\left(p-\frac{1}{2}\right)s_{i+1}+\cdots+Q_{p+1}^{(k)}\left(\frac{1}{2}\right)s_{i+p}\right\}$$

$$(*)\qquad =Q_{p+1}\left(p+1-\frac{1}{2}\right)s_i^{(k)}+Q_{p+1}\left(p-\frac{1}{2}\right)s_{i+1}^{(k)}+\cdots+Q_{p+1}\left(\frac{1}{2}\right)s_{i+p}^{(k)} \quad ([2]).$$

Here $s_i=s(ih)$ and $s_i^{(k)}=s^{(k)}(ih)$.

From now on, let p and k be odd and even integers, respectively. Since k is even, in virtue of the properties:

$$Q_{p+1}(x)\equiv Q_{p+1}(p+1-x)$$
$$Q_{p+1}(x)\equiv 0 \qquad \text{for } x\leq 0,\ x\geq p+1,$$

we have

$$c_j^{(l)}=Q_{p+1}^{(l)}\left(p+\frac{1}{2}-j\right)-Q_{p+1}^{(l)}\left(p+\frac{3}{2}-j\right)+\cdots$$
$$=(-1)^{j-p}\left\{Q_{p+1}^{(l)}\left(p+\frac{1}{2}\right)-Q_{p+1}^{(l)}\left(p-\frac{1}{2}\right)+\cdots\right\}$$

for $l=0,\ k$ and $j=p,\ p+1,\ \cdots$.

Since p is odd, in virtue of the property:

$$Q_{p+1}^{(l)}\left(p+\frac{1}{2}-j\right)=Q_{p+1}^{(l)}\left(j+\frac{1}{2}\right)$$

for $l=0,\ k$ and $j=p,\ p+1,\ \cdots,$