61. A Result on the Siegel-Ramachandra Class Invariant over Imaginary Quadratic Fields

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Let K be an imaginary quadratic field embedded in the complex number field $\boldsymbol{C}$, and let $\mathfrak{f}\neq(1)$ be an integral ideal of K. For each element C of the ray class group $Cl(\mathfrak{f})$ modulo $\mathfrak{f}$ the ray class invariant $\phi_{\mathfrak{f}}(C)$ is introduced by Siegel and Ramachandra (cf. Robert [5] and Stark [7]). The definition will be explained in the text. Let $H(\mathfrak{f})$ be the ray class field modulo $\mathfrak{f}$ and f the smallest positive integer contained in $\mathfrak{f}$. Then it is known that $(\phi_{\mathfrak{f}}(C)/\phi_{\mathfrak{f}}(C'))^{e(H(\mathfrak{f}))/e(K)}$ $(C, C'\in Cl(\mathfrak{f}))$ is the $(12f)$-th power of a unit of $H(\mathfrak{f})$ (Gillard and Robert [1]). Here $e(H(\mathfrak{f}))$ and $e(K)$ are the numbers of roots of unity contained in $H(\mathfrak{f})$ and K respectively. In this paper we describe a $(12f)$-th root of $(\phi_{\mathfrak{f}}(C)/\phi_{\mathfrak{f}}(C'))^{e(H(\mathfrak{f}))/e(K)}$ contained in $H(\mathfrak{f})$ explicitly by special values of the Siegel functions and determine the behavior under Artin automorphisms. The result is then useful to calculate class numbers of abelian extensions of K by the method of Gras and Gras [2].

§1. **Preliminaries.** Let $\mathfrak{f}\neq(1)$ be an integral ideal of K. The ideal $\mathfrak{f}$ is uniquely decomposed into two factors $\mathfrak{f}_a$, $\mathfrak{f}_b$ as follows:

$$\mathfrak{f}=\mathfrak{f}_a\mathfrak{f}_b,\quad \mathfrak{f}_a=\bar{\mathfrak{f}}_a,\quad (\mathfrak{f}_b,\bar{\mathfrak{f}}_b)=1.$$

Here the bar indicates the complex conjugation. Take an integral basis $\{\omega, 1\}$ $(\mathrm{Im}\,(\omega)>0)$ of the ring $\mathfrak{o}$ of integers of K. We fix such an ω throughout this paper. The next lemma is fundamental in the formulation and the proof of our results.

Lemma. *Let f_b be the smallest positive integer contained in $\mathfrak{f}_b$. Then there exists a rational integer a satisfying the following condition: For an arbitrary element x of $\mathfrak{f}_b$ the congruence*

$$a\,\mathrm{tr}\,(x)\equiv \mathrm{Im}\,(x)/\mathrm{Im}\,(\omega) \mod f_b$$

holds, where $\mathrm{tr}\,(\cdot)$ *is the trace map from K to the rational number field $\boldsymbol{Q}$.*

We fix such an integer a. For an algebraic number field H of finite degree denote by $e(H)$ the number of roots of unity contained in H. Put $\delta=e(H(1))/e(K)$, where $H(1)$ is the Hilbert class field of K. The integer δ is a divisor of 6. We consider the following condition (♯) concerning an ideal (not necessarily integral) $\mathfrak{a}$ of K:

(♯) $\mathfrak{a}$ is prime to $6\mathfrak{f}$ and $N(\mathfrak{a})\equiv 1 \bmod (12/\delta)$.

Here $N(\mathfrak{a})$ is the absolute norm of $\mathfrak{a}$ and the congruence is considered