# 59. On an Anderson-Anderson Problem

By Ryûki MATSUDA
Department of Mathematics, Faculty of Science, Ibaraki University

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1983)

§ 1. **Introduction.** Let $R$ be a commutative integral domain with quotient field $K$. For nonzero fractional ideals $I$ and $J$, we define $I:J=\{x \in R; xJ \subset I\}$. We will denote $\{x \in K; xI \subset R\}$ by $I^{-1}$, and $(I^{-1})^{-1}$ by $I_v$. We will say that $I$ is a *divisorial ideal* or *v-ideal* if $I=I_v$. I is a $v$-ideal of *finite type* if $I=J_v$ for some finitely generated fractional ideal $J$. By a *graded domain* $R=\oplus_{\alpha \in \Gamma} R_\alpha$, we mean an integral domain $R$ graded by an arbitrary torsionless grading monoid $\Gamma$, i.e., a commutative cancellative monoid, written additively, such that the quotient group $\langle \Gamma \rangle$ generated by $\Gamma$ is a torsion-free abelian group. (A general reference on torsionless grading monoids and $\Gamma$-graded rings is [5].) For a fractional ideal $I$ of a $\Gamma$-graded integral domain $R=\oplus_{\alpha \in \Gamma} R_\alpha$, $I^*$ will denote the fractional ideal generated by the homogeneous elements of $I$. Let $x \in R$, with $x=x_1+x_2+\cdots+x_n$, where $x_i \in R_{\alpha_i}$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. We then define the content of $x$, denoted by $C(x)$, to be $(x_1, x_2, \cdots, x_n)$. One of the most important examples of a $\Gamma$-graded integral domain is the semigroup ring $R[X;\Gamma]$. Here $R[X;\Gamma]=R[\{X^g; g \in \Gamma\}]$ with $X^g X^h = X^{g+h}$. $R[X;\Gamma]$ is $\Gamma$-graded in the natural way with $\deg(X^g)=g$. In [1], D. D. Anderson-D. F. Anderson studied $v$-ideals and invertible ideals of a $\Gamma$-graded domain $R$. Specifically in § 3 they gave necessary and sufficient conditions for an integral $v$-ideal of $R$ to be homogeneous whenever it contains a nonzero homogeneous element, proving the following result.

**Theorem** ([1], Theorem 3.2). *Let $R=\oplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and suppose $S=\{$nonzero homogeneous elements of $R\} \neq \phi$. The following statements are equivalent.*

(1) *For $r \in S$ and $x \in R$, $(r):(x)$ is homogeneous.*

(2) *If $I$ is an integral v-ideal of $R$ with nonzero $I^*$, then $I$ is homogeneous.*

(3) *If $I$ is an integral v-ideal of $R$ of finite type with nonzero $I^*$, then $I$ is homogeneous.*

(4) *$C(xy)_v=(C(x)C(y))_v$ for all nonzero $x, y \in R$.*

(5) *For each nonzero $x \in R$, $xR_S \cap R = xC(x)^{-1}$.*

(6) *If $I$ is an integral v-ideal of $R$ of finite type, then $I=qJ$ for some $q \in R_S$ and some homogeneous integral v-ideal $J$ of $R$ of finite type.*