6. Global Solutions of the Nonlinear Schrödinger Equation in Exterior Domains

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§ 1. Introduction and theorem. We consider the following initial boundary value problem for the nonlinear Schrödinger equation in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$:

\begin{align}
  i \frac{\partial u}{\partial t} &= \Delta u + \lambda |u|^p u \quad \text{in } [0, \infty) \times \Omega, \\
  u(0, x) &= u_0(x), \\
  u|_{\partial \Omega} &= 0.
\end{align}

Here $\lambda$ is a real constant and $p$ is an even integer with $p \geq 2$. The domain $\Omega$ is the exterior of a compact set in $\mathbb{R}^n$, $n \geq 3$, with the smooth boundary $\partial \Omega$. In the present paper we shall prove that Problem (1.1)-(1.3) has a unique global solution for small initial data under a certain assumption on the shape of $\Omega$, which indicates that $\Omega$ is "non-trapping" in the sense of Vainberg [2] and Rauch [3].

For the Cauchy problem, namely the case of $\Omega = \mathbb{R}^n$, the above problem has been extensively studied. For the exterior problem, however, we know only the work of Brézis and Gallouet [1]. In [1] they treated Problem (1.1)-(1.3) only for the case of $n = 2$.

We shall first give some notations. For an open set $D$ in $\mathbb{R}^n$, let $H^s(D)$, $H^s_0(D)$, $L^q(D)$, $L^q_0(D)$ and $C^k(D)$ denote the standard function spaces. We shall fix $R > 0$ such that $\partial \Omega \subset \{x \in \mathbb{R}^n; |x| < R\}$. For any $r \geq R$, we denote the set $\{x \in \Omega; |x| < r\}$ by $\Omega_r$. We shall often abbreviate $\left( \frac{\partial}{\partial x} \right)^\alpha$ and $\left( \frac{\partial}{\partial t} \right)^j$ to $\partial_x^\alpha$ and $\partial_t^j$ respectively, where $\alpha$ is a multi-index and $j$ is a nonnegative integer. For $a \in \mathbb{R}^n$ we denote by $[a]$ the greatest integer that is not larger than $a$.

Let $G = G(t, x, x_0)$ be the Green function for the following problem:

\begin{align}
  (\partial^2 / \partial t^2 - \Delta)G &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
  \lim_{t \to +0} \frac{\partial^j G}{\partial t^j} &= \begin{cases} 
  0, & j = 0, \\
  (\delta(x-x_0), & j = 1, \\
  G|_{x=x_0} &= 0,
\end{cases}
\end{align}

where $x_0$ is an arbitrary point of $\Omega$. For any $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ we define $f(t, x, x_0)$ by

\begin{align}
  f(t, x, x_0) = \int_{\Omega} \psi(x) G(t, x, x_0) \, dx.
\end{align}