50. Toda Lattice Hierarchy. I

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(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1983)

0. Introduction. In this note we shall consider a hierarchy (a sequence of mutually commuting higher evolutions) for the two dimensional Toda lattice (TL)

(1) $$\partial_{x_1}\partial_{y_1}u(s)=e^{u(s)-u(s-1)}-e^{u(s+1)-u(s)},$$

where $u(s)=u(s\,;x_1,y_1)$, $\partial_{x_1}=\partial/\partial x_1$, $\partial_{y_1}=\partial/\partial y_1$ and s runs over Z, the totality of integers. The Toda lattice is, together with the Korteweg-de Vries (KdV) equation, one of the most classical and important completely integrable systems. Several varieties of methods have been developed to reveal its profound mathematical structure.

The present work is inspired by the recent developments in the study of the Kadomtsev-Petviashvili (KP) equation and its hierarchies [1], [2]. Our method enables us to investigate the infinite Toda lattice in an extremely clear and algebraic framework.

Let us begin with the following observations: In the periodic case the so called zero curvature representation of (1) is given in [3]. In the general case (1) is represented in the form

(2) $$\partial_{y_1}B_1-\partial_{x_1}C_1+[B_1,C_1]=0,$$

where the symbol [,] denotes the commutator, and B_1, C_1 are the matrices $B_1=(\delta_{i,j-1})_{ij\in Z}+(\partial_{x_1}u(i)\delta_{i,j})_{ij\in Z}$, $C_1=(e^{u(i)-u(i-1)}\delta_{i,j+1})_{ij\in Z}$ of size $Z\times Z$.

If the τ function $\tau(s)=\tau(s\,;x_1,y_1)$ is introduced by

(3) $$u(s)=\log\,[\tau(s+1)/\tau(s)],$$

(1) is transformed into the bilinear equation of the Hirota type

(4) $$D_{x_1}D_{y_1}\tau(s)\cdot\tau(s)+2\tau(s+1)\tau(s-1)=0,$$

where $D_{x_1}D_{y_1}$ is one of Hirota's D-operators which are in general defined, for a linear differential operator $P(\partial_t)$, by

(5) $$P(D_t)f(t)\cdot g(t)=P(\partial_{t'})f(t+t')g(t-t')|_{t'=0}.$$

By use of the function $\tau'(s)=e^{x_1y_1}\tau(s)$, (4) is rewritten into the original form of Hirota's bilinearization [4]

(6) $$D_{x_1}D_{y_1}\tau'(s)\cdot\tau'(s)+2\tau'(s+1)\tau'(s-1)-2\tau'(s)^2=0.$$

The N soliton solution to (6) is presented in [4]. A parametrization of the τ function $\tau'(s)$ in terms of Clifford operators is discussed in [5].

We shall extend these observations to a hierarchy for (1).

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