# 46. A Generalization of Gauss' Theorem on Arithmetic-Geometric Means

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§ 1. **Introduction and methods.** With each continuous map $f$: $\boldsymbol{R}^n \to \boldsymbol{R}^m$ we associate an entire function $f^\sharp(z)$ given by

$$f^\sharp(z)=\int_{S^{n-1}} e^{zN(f(x))} d\omega_{n-1}.^{*)}$$

We shall assume throughout that

(1.1) $$f(x)\neq 0 \quad \text{for all } x \in S^{n-1},$$

hence $N(f(x))>0$ on $S^{n-1}$. When it is so, the integral

(1.2) $$\Gamma(f\,;s)=\int_0^\infty t^{s-1}f^\sharp(-t)dt$$

represents a holomorphic function for $\sigma=\mathrm{Re}\, s>0$. We have

(1.3) $$\Gamma(f\,;s)=\Gamma(s)K(f\,;s)$$

where $\Gamma(s)$ is the usual gamma function and

(1.4) $$K(f\,;s)=\int_{S^{n-1}} N(f(x))^{-s} d\omega_{n-1}.$$

By (1.1), $K(f\,;s)$ is entire and (1.3) yields the meromorphic continuation of $\Gamma(f\,;s)$ onto $\boldsymbol{C}$.

When $n=m=2$, $f(x)=(ax_1, bx_2)$, $0<a\leqq b$ and $s=1/2$, our $K(f\,;s)$ becomes the complete elliptic integral:

$$K\left(f\,;\frac{1}{2}\right)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}.$$

Gauss proved, by means of quadratic transformations of theta series,

(G) $$K\left(f\,;\frac{1}{2}\right)=K\left(f_1\,;\frac{1}{2}\right), \qquad f_1(x)=(a_1x_1, b_1x_2)$$

where $a_1=\sqrt{ab}$, $b_1=(a+b)/2$.**) The repeated application of (G) yields immediately the relation $K(f\,;1/2)=M(a,b)^{-1}$ where $M(a,b)$ means the arithmetic-geometric of $a$, $b$.

In this paper, we shall generalize (G) for our $K(f\,;s)$ defined by (1.4) when $n=m=2p$, $p>\sigma=\mathrm{Re}\, s>(p-1)/2$ and $f(x)=(ax_1, \cdots, ax_p, bx_{p+1}, \cdots, bx_{2p})$. The proof depends on the fact that, under the assumptions, $K(f\,;s)$ can be expressed as a hypergeometric series via

*) We denote by $\langle x, y\rangle$ the standard inner product in $\boldsymbol{R}^n$. We put $Nx=\langle x, x\rangle$. The unit sphere is $S^{n-1}=\{x\in\boldsymbol{R}^n; Nx=1\}$. We denote by $d\omega_{n-1}$ the volume element of $S^{n-1}$ such that the volume of $S^{n-1}$ is 1.

**) See [1] p. 352. See also [7] § 9 and [8] p. 269.