41. Null Strings and Null Solutions of Maxwell's Equations

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1. The purpose of this note is to establish a relation between null strings and null solutions of Maxwell's equations (null electromagnetic fields). We will express Plücker's coordinates of the 2-dimensional surface swept out by a null string by means of a null electromagnetic field. The detailed proof will be given elesewhere.

Let (M, g) be Minkowski space with the affine coordinates x^μ, $\mu=0,1,2,3$. Throughout this note we adopt the summation convention. Namely, we suppress the summation sign every time that the summation has to be done on an index which appears twice in the term. To raise or lower indices, we use the formulas $X_\mu=g_{\mu\nu}X^\nu$ and $X^\mu=g^{\mu\nu}X_\nu$, where $g_{\mu\nu}$ are the components of the metric g and the matrix $(g^{\mu\nu})$ is the inverse of the matrix $(g_{\mu\nu})$.

A string is an extremal, i.e., a solution of the Euler-Lagrange equations, of the variational problem for 2-dimensional surfaces $\Sigma: x^\mu(\tau^1, \tau^2)$ in M:

$$\delta\left(\int L d\tau^1 d\tau^2\right)=0$$

with Lagrangean $L=v_{\mu\nu}v^{\mu\nu}$, where we define Plücker's coordinates $v^{\mu\nu}$ of the string Σ by

$$v^{\mu\nu}=(\partial x^\mu/\partial\tau^1)(\partial x^\nu/\partial\tau^2)-(\partial x^\nu/\partial\tau^1)(\partial x^\mu/\partial\tau^2).$$

A string on which $L=0$ is said to be null. In Minkowski space, Maxwell's equations take the Lorentz invariant form $dF=0=d{*}F$ for differential 2-forms F, where $*$ is Hodge operator. If a solution F of Maxwell's equations satisfies $g(F, F)=0=g(F, {*}F)$, we call it a null electromagnetic field.

Our main theorems are as follows.

Theorem 1. *Let a 2-dimensional surface $\Sigma: x^\mu(\tau^1, \tau^2)$ in Minkowski space be an analytic null string and let $x_{(0)}$ be a point on Σ. Then on some neighbourhood U_0 of $x_{(0)}$ in Minkowski space, a null electromagnetic field $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ with a following property exists:*

(1) $\quad v^{\mu\nu}(\tau^1, \tau^2)=F^{\mu\nu}(x(\tau^1, \tau^2)), \qquad x(\tau^1, \tau^2)=(x^\mu(\tau^1, \tau^2))\in U_0.$

Theorem 2. *Let $F=F_{\mu\nu}dx^\mu\wedge dx^\nu$ be a null electromagnetic field. If a 2-dimensional surface $\Sigma: x^\mu(\tau^1, \tau^2)$ in Minkowski space satisfies* (1), *then Σ is a null string.*