40. *Ergodic Theorems for Semigroups of Operators on a Grothendieck Space*

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1. Introduction. A theorem of Atalla [1] states that the Cesàro means $\{(1/n)(T+\cdots+T^n)\}$ of a linear contraction T on a Grothendieck space X converge strongly if and only if the weak* closure and the strong closure of the range of T^*-I^* (the dual operator of $T-I$) coincide. The main purpose of the present paper is to prove an analogue of this theorem for semigroups of linear operators.

Throughout the paper X is a Grothendieck space, i.e., weak* sequential convergence in the dual space X^* is equivalent to weak sequential convergence (cf. [3]), and $\{T(t)\}_{t>0}$ is a locally integrable semigroup of linear operators on X. By this we mean that for each $x \in X$ $T(\cdot)x$ is strongly measurable on $(0, \infty)$ and $\int_0^t \|T(s)x\| ds < \infty$ for every $t \in (0, \infty)$. Then the Bochner integral $\int_0^t T(s)xds$ exists for all $x \in X$. Since $T(\cdot)$ is strongly continuous on $(0, \infty)$ (see [4, p. 616]), this integral is also an improper Riemann integral.

Let $S(t)$ denote the operator on X such that $S(t)x=\int_0^t T(s)xds$ for all $x \in X$. Then $S(t)$ is a continuous linear operator (see [4, p. 685]). The ergodic theory is concerned with the existence of $\lim_{t\to\infty} t^{-1}S(t)x$. When the limit exists strongly for all x in X, $T(\cdot)$ is said to be strongly ergodic.

First we specify some notations. P will stand for the map which sends x to the strong limit $\text{s-lim}_{t\to\infty} t^{-1}S(t)x$ whenever the limit exists; its domain $D(P)$ is the set of all x for which the limit exists. Similarly, Q is the map in X^* determined by the weak* limits $\text{w*-lim}_{t\to\infty} t^{-1}S^*(t)x^*$. Also we shall use the following notations:

$$\boldsymbol{F}=\bigcap_{t>0} N(T(t)-I); \qquad \boldsymbol{F}^*=\bigcap_{t>0} N(T^*(t)-I^*);$$

$$\boldsymbol{R}=\text{span}\left\{\bigcup_{t>0} R(T(t)-I)\right\}; \qquad \boldsymbol{R}^*=\text{span}\left\{\bigcup_{t>0} R(T^*(t)-I^*)\right\},$$

where $N(L)$ and $R(L)$ are the null space and the range of an operator L.

We shall prove the following theorems.

Theorem 1. *Let $T(\cdot)$ be a locally integrable semigroup of operators on a Grothendieck space X. Assume that*