4. The Order of Unstable Manifold of some Algebraic Plane Transformation

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We consider the transformation

$$f(x, y)=(y+cx(1-x), x), \qquad c>0. \tag{1}$$

(See [1] and [2].)
The entire solution of the functional equation:

$$\begin{aligned} &g(0)=0, \\ &g(\lambda t)=f(g(t)) \quad (\lambda>0), \\ &g(t)=(\alpha(t), \beta(t)), \end{aligned} \tag{2}$$

is called unstable manifold of the transformation f through origin.

The order ρ of g is defined by the following formula:

$$\rho=\limsup_{r\to\infty} \log\log M(r)/\log r$$

where $M(r)$ is the maximum value of $|\alpha(t)|$ on $|t|=r$.

Proposition 1. $M(r)=-\alpha(-r)$.

Proof. From (2) α satisfies

$$\alpha(\lambda^2 t)=\alpha(t)+c\alpha(\lambda t)(1-\alpha(\lambda t)). \tag{3}$$

Since $\alpha=\sum \alpha_n t^n$, $\alpha_1=1$, we deduce that

$$\lambda^2-c\lambda-1=0 \ (\lambda>0) \quad \text{and} \quad \alpha_n=\frac{-c\lambda^n}{\lambda^{2n}-c\lambda^n-1}\sum_{i_1+i_2=n}\alpha_{i_1}\alpha_{i_2}.$$

From the above identity, we obtain by induction $\alpha_{2n-1}>0$ and $\alpha_{2n}<0$. Consequently, we get

$$M(r)=-\alpha(-r).$$

Theorem 1. $\rho=\log 2/\log\lambda$.

Proof. Since

$$-\alpha(-r)=-\alpha(-r/\lambda^2)-c\alpha(-r/\lambda)+c\alpha(-r/\lambda)^2>c\alpha(-r/\lambda)^2,$$

we get $\rho\geq\log 2/\log\lambda$. Conversely, we can derive the next inequality.

$$\begin{aligned} -\alpha(-r)&=-\alpha(-r/\lambda^2)-c\alpha(-r/\lambda)+c\alpha(-r/\lambda)^2 \\ &<-\alpha(-r/\lambda^2)+k_1\alpha(-r/\lambda)^2 \\ &<k_2(\underbrace{\alpha(-r/\lambda)^2+\alpha(-r/\lambda^3)^2+\cdots+\alpha(-r/\lambda^{2n-1})^2}_{n}) \\ &<k_2 n\alpha(-r/\lambda)^2, \end{aligned}$$

where n is approximated by $\log r/2\log\lambda$, and k_1 and k_2 do not depend on r. This implies

$$\rho\leq\log 2/\log\lambda.$$

Combining the above relations, we get the consequence.