31. Short Term Consistency Relations for Doubly Polynomial Splines

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By making use of the B-spline $Q_{m+1}(x)$:

$$Q_{m+1}(x)=(1/m!)\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(x-i)_+^m$$

where

$$(x-i)_+^m=\begin{cases}(x-i)^m & \text{for } x\geq i\\ 0 & \text{for } x<i,\end{cases}$$

we consider a quartic spline $s(x)$ of the form:

$$s(x)=\sum_{i=-4}^{n-1}\alpha_i Q_5(x/h-i), \qquad nh=1.$$

Then the following short term consistency relation has been obtained by Usmani ([6]):

$$(*) \qquad (s_{i+1}-2s_i+s_{i-1})=(h^2/12)(s''_{i+1}+10s''_i+s''_{i-1})$$

where $s_i=s(ih)$ and $s''_i=s''(ih)$. The above relation has been generalized for even degree polynomial splines ([3]). For odd degree polynomial splines, we also have short term consistency relations at mid-points ([4]). For example, let $s(x)$ be a cubic, then

$$(**) \qquad (s_{i+3/2}-2s_{i+1/2}+s_{i-1/2})=(h^2/24)(s''_{i+3/2}+22s''_{i+1/2}+s''_{i-1/2})$$

where $s_{i+1/2}=s((i+1/2)h)$ and $s''_{i+1/2}=s''((i+1/2)h)$.

In the present paper we shall generalize the above relations $(*)$ and $(**)$ for doubly polynomial splines.

Let $s(x, y)$ be a polynomial spline of the form:

$$s(x,y)=\sum_{i,j=-m}^{n-1}\alpha_{i,j}Q_{m+1}(x/h-i)Q_{m+1}(y/h-j).$$

Then we have

Theorem 1. *If m is even and $k, l\ (\leq m-2)$ are also even, we have*

$$\sum_{i,j=0}^{m-2}c_{i,j}^{(k,l)}s_{i,j}=h^{k+l}\sum_{i,j=0}^{m-2}c_{i,j}^{(0,0)}s_{i,j}^{(k,l)}$$

where

$$s_{i,j}^{(k,l)}=\frac{\partial^{k+l}}{\partial x^k\partial y^l}s(ih, jh)$$

$$c_{i,j}^{(k,l)}=\{Q_{m+1}^{(k)}(m-i)-Q_{m+1}^{(k)}(m-i+1)+\cdots\}$$
$$\times\{Q_{m+1}^{(l)}(m-j)-Q_{m+1}^{(l)}(m-j+1)+\cdots\}.$$

Proof. The following m^2-term consistency relation holds: