103. *Zeros, Primes and Rationals*

By Akio FUJII
Department of Mathematics, Rikkyo University

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§ 1. **Introduction.** The connections between the primes and the zeros of the Riemann zeta function $\zeta(s)$ have been expressed in the explicit formulae since Riemann. It is Landau who showed some arithmetical connection between them; on the Riemann Hypothesis,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{0<\gamma<T} e^{ia\gamma}=\begin{cases}-\dfrac{\log p}{2\pi p^{k/2}} & \text{if } a=k\log p\\ 0 & \text{otherwise,}\end{cases}$$

where γ runs over the positive imaginary parts of the zeros of $\zeta(s)$, p is a prime and k is an integer $\geqq 1$. Here we remark the following arithmetical connection between the zeros and the rationals which we have remarked in [3] and [4].

Theorem 1. *Let α be a positive number and b be a real number $\leqq 1$. Then on the Riemann Hypothesis,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{2\pi e\alpha<\gamma\leqq T} e^{i\gamma(\log(\gamma/2\pi e\alpha))^b}=\begin{cases}-\dfrac{e^{i\pi/4}}{2\pi}C(\alpha) & \textit{if } b=1 \textit{ and } \alpha \textit{ is rational}\\ 0 & \textit{otherwise,}\end{cases}$$

where $C(\alpha)=\mu(k)/(\sqrt{\alpha}\,\varphi(k))$ with the Möbius function $\mu(k)$ and the Euler function $\varphi(k)$ when $\alpha=l/k$, l and k are integers $\geqq 1$ and $(l,k)=1$.

In fact, we have proved a theorem on $\sum_{C<\gamma\leqq T} e^{if(\gamma)}$ for more general f without assuming any unproved hypothesis and given a different proof to the author's previous result (cf. [2]) which states that $f(\gamma)$ is uniformly distributed mod one, where $f(\gamma)$ may be, for example, $\gamma\log\gamma/\log\log\log\gamma$, $\gamma(\log\gamma)^b$ with $b<1$ and γ. Landau's theorem and Theorem 1 can be extended to Dirichlet L-functions $L(s,\chi)$ and these have also q-analogues (cf. [4]). We state here only a q-analogue of Theorem 1. Let $\sum'_\chi$ denote the summation over all non-principal characters $\chi \bmod q$. We suppose, for simplicity, that q runs over the primes. Let $\gamma(\chi)$ denote an imaginary part of the non-trivial zeros of $L(s,\chi)$. Then our q-analogue of Theorem 1 can be stated as follows.

Theorem 2. *Let η be an integer, α be a positive number and b be a real number $\leqq 1$. We assume the generalized Riemann Hypothesis and suppose that $T=T(q)$ satisfies $q^\nu(\log q)^B\ll T\ll q^A$, where ν is a constant depending on η, $B>B_0$ and A is an arbitrarily large constant.*